A functional representation approach to vector lattice covers for spaces of compact operators

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Motivation (1)

Let X, Y be Archimedean vector lattices and consider the space $\mathrm{L}^{\mathrm{r}}(X, Y)$ of regular linear operators with the cone $\mathrm{L}_+(X, Y) := \{T \in \mathrm{L}(X, Y); T[X_+] \subseteq Y_+\}$.

- If Y is Dedekind complete:
  $\mathrm{L}^{\mathrm{r}}(X, Y)$ is a Dedekind complete vector lattice
- In general:
  $\mathrm{L}^{\mathrm{r}}(X, Y)$ is an Archimedean directed ordered vector space

  Problem: Determine the set of operators that have a modulus.

Motivation (2)

Let Z be an Archimedean directed ordered vector space.
Determine $M := \{z \in Z;\ |z| := \sup\{z, -z\} \text{ exists}\}$.

Proposition (K., Stennder, van Gaans, 2021)

For $z \in Z$, the following are equivalent.

(i) $|z|$ *exists.*

(ii) *There are $z_1, z_2 \in Z_+$ with $z_1 \perp z_2$ and $z = z_1 - z_2$.*

- We are going to define disjointness in ordered vector spaces.
- We use a vector lattice cover of Z to determine all disjoint elements (and, hence, all elements with modulus).

Ordered vector spaces

Let X be a (real) vector space. A partial order $\leq$ on X is called a vector space order if

(a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

(b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

The set $X_+ := \{x \in X; 0 \leq x\}$ is then a cone in X, i.e., $x, y \in X_+$, $\lambda \in [0, \infty)$ imply $\lambda x + y \in X_+$, and $X_+ \cap (-X_+) = \{0\}$.

X is then called an ordered vector space (ovs).

For X, Y being ovs and $T\colon X \to Y$ a linear operator, T is called bipositive if T is positive (i.e., $T[X_+] \subseteq Y_+$) and, for $x \in X$, $Tx \in Y_+$ implies $x \in X_+$.

Pre-Riesz spaces

An ordered vector spaces X is called a pre-Riesz space if there exist a vector lattice Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y, i.e., for every $y \in Y$ one has

$$y = \inf\{i(x);\ x \in X,\ i(x) \geq y\}.$$

(Y, i) is called a vector lattice cover.

Every vector lattice is a pre-Riesz space.

An ovs X is called Archimedean if, for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$, one has that $x \leq 0$.
X is directed if and only if X_+ is generating, i.e., $X = X_+ - X_+$.

Proposition

- *Every Archimedean directed ovs is a pre-Riesz space.*
- *Every pre-Riesz space is directed.*

If X, Y are ovs such that X is directed and Y is Archimedean, then $\mathrm{L}_+(X, Y) := \{T \colon X \to Y;\ T \text{ linear},\ T[X_+] \subseteq Y_+\}$ is a cone and $\mathrm{L}^{\mathrm{r}}(X, Y) := \mathrm{L}_+(X, Y) - \mathrm{L}_+(X, Y)$ is an Archimedean directed ovs, hence pre-Riesz.

Explicite construction of vector lattice covers of spaces of operators:

- $\mathrm{L}^{\mathrm{r}}(\ell_0^\infty, Y)$, where ℓ_0^∞ is the space of all finally constant sequences and Y is an Archimedean vector lattice [Wickstead, 2024]
- generalized version in [Starkey, Xanthos, 2025]

Hereby, the vector lattice cover is a space of operators, where the range space is Dedekind complete.

- The operator norm closure $\mathcal{C}(X, Y)$ of the finite rank operators within $\mathcal{L}(X, Y)$, where X and Y are appropriate ordered normed spaces such that $\mathcal{L}(X, Y)_+$ contains an order unit [van Gaans, Glück, K., 2025]

Vector lattice cover is $\mathrm{C}(\Omega)$ for Ω being a compact Hausdorff space.

Order unit spaces

Let X be an Archimedean ovs with order unit u, i.e., for every $x \in X$ there is a $\lambda \in (0, \infty)$ such that $-\lambda u \leq x \leq \lambda u$. By defining

$$\|\cdot\|_u : X \to [0, \infty), \quad x \mapsto \|x\|_u := \inf\{\lambda \in (0, \infty);\ -\lambda u \leq x \leq \lambda u\},$$

X is a normed space. X' denotes the (norm) dual space of X and $X'_+ := \{\varphi \in X';\ \varphi[X_+] \subseteq [0, \infty)\}$ the dual cone. The set

$$\Sigma = \{\varphi \in X'_+;\ \varphi(u) = 1\}$$

is a weakly-$*$ compact base of X'_+.

Note: Order unit spaces are directed.

Kadison's representation

Define

$$\Psi\colon X \to \mathrm{C}(\Sigma), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

- Ψ is linear, bipositive, and maps u to the constant-1 function.
- For every $x \in X$, the function $\Psi(x)$ is affine on Σ.

To obtain an order dense embedding, one has to consider

$$\Phi\colon X \to \mathrm{C}\left(\overline{\mathrm{ext}(\Sigma)}\right), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

[K.,Lemmens, van Gaans, 2013]

$\left(\mathrm{C}\left(\overline{\mathrm{ext}(\Sigma)}\right), \Phi\right)$ is a vector lattice cover of X.

Our assumptions:

Let X, Y be (non-zero) ordered normed spaces with closed cones. This implies that X and Y are Archimedean.

Consider the space $\mathcal{L}(X, Y)$ of linear norm bounded operators.

Let X_+ be total, i.e., $\overline{X_+ - X_+} = X$.
X is total if and only if $\mathcal{L}(X, Y)_+$ is a cone.

Every U in the interior of $\mathcal{L}(X, Y)_+$ is an order unit.

Theorem (van Gaans, Glück, K., 2025)

The following are equivalent:

1. $\mathcal{L}(X, Y)_+$ *has non-empty interior in* $\mathcal{L}(X, Y)$.
2. *The cone* Y_+ *has non-empty interior in* Y *and there exists an equivalent norm on* X *which is additive on* X_+.

In this case, the interior of $\mathcal{L}(X, Y)_+$ contains a rank-1 operator:

For every interior point y_0 of Y_+ and every interior point x_0' of X_+', the operator $y_0 \otimes x_0'$ is an interior point of $\mathcal{L}(X, Y)_+$.

Disjointness

For $x, y \in X_+$, define $x \perp y$ whenever $x \wedge y = 0$.

If X is a vector lattice: For $x, y \in X$, $x \perp y$ whenever $|x| \wedge |y| = 0$.
Equivalent: $|x + y| = |x - y|$.

If X is an ovs [van Gaans, K. 2006]:

$$x \perp y \quad :\Longleftrightarrow \quad \{x + y, -(x + y)\}^{\mathrm{u}} = \{x - y, -(x - y)\}^{\mathrm{u}}$$

For $M \subseteq X$, M^{d} denotes the disjoint complement of M.
M is called a band, if $M = M^{\mathrm{dd}}$.

Proposition (K., Stennder, van Gaans, 2021)

Let Z be a pre-Riesz space. The set of elements in Z that possess a modulus in Z equals

$$\bigcup_{B \subseteq Z \text{ band}} \left(B_+ - B_+^{\mathrm{d}}\right).$$

Disjointness under embedding

Proposition (van Gaans, K., 2006)

Let X be a pre-Riesz space and (Y, i) a vector lattice cover of X. Then one has, for every $x, y \in X$,

$$x \perp y \iff i(x) \perp i(y).$$

Order denseness is needed for '$\Longrightarrow$'.

If $Y = \mathrm{C}(\Omega)$, disjointness is pointwise, and it is sufficient to consider a dense subset of Ω.

Modification of Kadison's embedding:

Theorem (van Gaans, Glück, K., 2025)

Let $Z \neq \{0\}$ be an ordered normed space whose cone Z_+ has an interior point z_0 and let $S \subseteq Z'$ be a subset with the following properties:

1. *One has $\langle s, z_0 \rangle = 1$ for all $s \in S$.*
2. *Every element of S is an extremal vector of Z'_+.*
3. *The set S determines positivity (i.e., for every $z \in Z$, one has $z \geq 0$ whenever $s(z) \geq 0$ for all $s \in S$).*

Endow the weak-$*$ closure $\overline{S}$ with the weak-$*$ topology and consider

$$\Phi \colon X \to \mathrm{C}(\overline{S}), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

Then $(\mathrm{C}(\overline{S}), \Phi)$ is a vector lattice cover of Z.

$z_1 \perp z_2$ in $Z \iff$ for every $s \in S$, one has $s(z_1) = 0$ or $s(z_2) = 0$.

Question/Problem: Extremals in the cone of operators ??

$\mathcal{C}(X, Y)$ - closure of the space of finite rank operators in $\mathcal{L}(X, Y)$.

For $x \in X$ and $y' \in Y'$, define $y' \otimes x \in \mathcal{C}(X, Y)'$ by

$$(y' \otimes x)(T) := y'(Tx)$$

for all $T \in \mathcal{C}(X, Y)$.

Theorem (vG, Gl, K., 2025)

Let X and Y be ordered normed spaces with the following properties:

1. *The cone X_+ is total and normal, and every extremal vector x of X_+ is also extremal in the bidual cone X''.*
2. *The cone Y_+ is total.*

For non-zero vectors $x \in X_+$ and $y' \in Y'_+$, the functional $y' \otimes x \in \mathcal{C}(X, Y)'_+$ is extremal in $\mathcal{C}(X, Y)'_+$ if and only if x is extremal in X_+ and y' is extremal in Y'_+.

Theorem (vG, Gl, K, 2025)

Let X, Y be (non-zero) ordered normed spaces such that:

1. *The cone X_+ is total, there exists an equivalent norm on X that is additive on X_+, and every extremal vector of X_+ is also extremal in the bidual cone X''_+. Moreover, the convex hull of the extremal vectors in X_+ is dense in X_+.*
2. *The cone Y_+ has non-empty interior.*

For an interior point y_0 of Y_+ and an interior point x'_0 of X'_+, define

$$S := \{ y' \otimes x : \ x \text{ is extremal in } X_+,\ y' \text{ is extremal in } Y'_+, \\ \text{and } \langle y', y_0 \rangle \langle x'_0, x \rangle = 1 \} \subseteq \mathcal{C}(X, Y)'.$$

Then $(\mathrm{C}(\overline{S}), \Phi)$ is a vector lattice cover of $\mathcal{C}(X, Y)$.

Examples:

- If X, Y are finite-dimensional ordered vector spaces with closed and generating cones, then the assumptions of the theorem are satisfied.
  For polyhedral cones: [Schneider, Vidyasagar, 1970]
- The sequence space $X := \ell^1$ with its usual norm and the componentwise order satisfies assumption 1 of the theorem.
- Let H be a Hilbert space. The space X of self-adjoint trace class operators on H, endowed with the *Loewner cone* of operators $A \in X$ that satisfy $(v|Av) \geq 0$ for all $v \in H$, satisfies assumption 1 of the theorem.
  The extremal vectors of X_+ are precisely the strictly positive multiples of the rank-1 projections on H.

- Let X be a real Banach space, let $x_0 \in X$ and $x_0' \in X'$ be such that $\langle x_0', x_0 \rangle = 1$ and endow X with the *centered cone*

$$X_+ := \{x + rx_0 \colon \ x \in \ker x_0', \ r \geq \|x\|\}.$$

Then X_+ is a closed cone with non-empty interior and there exists an equivalent norm on X that is additive on X_+ [Glueck, 2016].

If the space X is reflexive, then X satisfies assumption 1 of the theorem.

Corollary

In the setting of the above theorem, two operators $T_1, T_2 \in \mathcal{C}(X, Y)$ are disjoint if and only if, for all extremal vectors $x \in X_+$ and $y' \in Y'_+$, one has $\langle y', T_1 x\rangle = 0$ or $\langle y', T_2 x\rangle = 0$.

- Bands in $\mathcal{C}(X, Y)$ are characterized by bisaturated subsets of $\overline{S}$. [K., Lemmens, van Gaans, 2015]
- The set of elements in $\mathcal{C}(X, Y)$ that possess a modulus equals

$$\bigcup_{B \subseteq \mathcal{C}(X,Y) \text{ band}} \left(B_+ - B_+^{\mathrm{d}}\right).$$

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