Some notes on quotient Banach lattices

Asghar Ranjbari   Leila Hasanzadeh

University of Tabriz

Positivity XII: 2-7 June 2023, Hammamet, Tunisia

Riesz spaces

• A Riesz space (or a vector lattice) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$ the supremum and the infimum of the set $\{x, y\}$ both exist in X.
• A vector x in a Riesz space X is called positive whenever $x \geq 0$ holds.
• The set of all positive vectors of X will be denoted by X^+, i.e., $X^+ = \{x \in X : \ x \geq 0\}$. The set X^+ of positive vectors is called the positive cone of X.

Definitions

• A net (x_α) in a Riesz space X is **order convergent** to $x \in X$, written as $x_\alpha \xrightarrow{o} x$, if there exists a net (y_β) possibly over a different index set, such that $y_\beta \downarrow 0$ and there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$.

• A net (x_α) in a Riesz space X is **unbounded order convergent** to $x \in X$, written as $x_\alpha \xrightarrow{uo} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in X^+$.

Definitions

• A subset I of a Riesz space is called **solid** whenever $|x| \leq |y|$ and $y \in I$ imply $x \in I$.

• A solid vector subspace of a Riesz space is referred to as an **ideal.**

• An order closed ideal is referred to as a **band.**

• For every element x of a Riesz space X, the **principal ideal generated** by x, X_x is defined as the following

$$X_x = \{y \in X : \ \exists\lambda > 0 \text{ with } |y| \leq \lambda|x|\}.$$

• The **principal band generated** by a vector x, B_x is given by

$$B_x = \{y \in X : \ |y| \wedge n|x| \uparrow |y|\}.$$

Definitions

• A Riesz space X is called **Archimedean** whenever $\frac{1}{n}x \downarrow 0$ holds in X for each $x \in X^+$.
• A vector $x > 0$ ($x \in X^+, x \neq 0$) in a Riesz space X is an **atom** if for any $u, v \in [0, x]$ with $u \wedge v = 0$, either $u = 0$ or $v = 0$.
• If X is an Archimedean Riesz space, then $x > 0$ in X is an atom if and only if X_x is one-dimensional.
• An Archimedean Riesz space X is called **atomic** if it is the band generated by its atoms.

P. Meyer-Nieberg, Banach Lattices, Universitext. Springer, Berlin (1991).

• A norm $\|.\|$ on a Riesz space is said to be a **lattice norm** whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.
• A Riesz space equipped with a lattice norm is known as a **normed Riesz space**.
• If a normed Riesz space is also norm complete, then it is referred to as a **Banach lattice**.
• A Banach lattice X is **order continuous** if $\|x_\alpha\| \to 0$ for any net (x_α) in X that order converges to 0.

• A net (x_α) in a Banach lattice X is **weak convergent** to $x \in X$, written as $x_\alpha \xrightarrow{w} x$, if $x^*(x_\alpha) \to x^*(x)$ in $\mathbb{R}$ for all $x^* \in X^*$ (norm dual of X).

• A net (x_α) in an Archimedean Riesz space X is **uniform convergent** to $x \in X$, written as $x_\alpha \xrightarrow{u} x$, if there exists $e \in X^+$ such that for every $\varepsilon > 0$ there exists α_0 such that $|x_\alpha - x| \leq \varepsilon e$ whenever $\alpha \geq \alpha_0$.

Quotient Banach lattices

Let X be a Banach lattice and let I be a normed closed ideal of X.
Let $\pi : X \to \frac{X}{I}$ that $\pi(x) = x + I$ for all $x \in X$, be the canonical mapping of X onto quotient space $\frac{X}{I}$.
π is linear, continuous and open.
We denote:

$$\dot{x} = \pi(x) = x + I,$$

for all $x \in X$.

Quotient Banach lattices

Recall that the quotient norm on $\frac{X}{I}$ defined as

$$\|\dot{x}\| = \|\pi(x)\| = \inf\{\|x - z\| : z \in I\} \qquad (\dot{x} \in \frac{X}{I}).$$

A relation $\leq$ is defined on $\frac{X}{I}$ by letting $\dot{x} \leq \dot{y}$ whenever there exist $x_1 \in \dot{x}$ and $y_1 \in \dot{y}$ with $x_1 \leq y_1$.
By the above norm and order relation, the quotient vector space $\frac{X}{I}$ is a Banach lattice and the canonical projection π of X onto $\frac{X}{I}$ is a lattice homomorphism.
It follows that $\dot{x^+} = (\dot{x})^+$, $\dot{x^-} = (\dot{x})^-$, $\dot{|x|} = |\dot{x}|$.

• The canonical projection of X onto $\frac{X}{I}$ is not necessarily order continuous.
• It is order continuous if and only if I is a band.

C. Aliprantis, O. Burkinshaw, Positive Operators. Springer, The Netherlands (2006).

unbounded norm convergence

A net (x_α) in a Banach lattice X is **unbounded norm (resp. unbounded absolute weak) convergent** to $x \in X$, written as $x_\alpha \xrightarrow{un} x$ (resp. $x_\alpha \xrightarrow{uaw} x$), if $|x_\alpha - x| \wedge u$ converges to zero in norm (resp. weak) for all $u \in X^+$.

Unbounded norm topology

Let X be a Banach lattice. For every $\varepsilon > 0$ and non-zero $u \in X^+$, put

$$V_{\varepsilon,u} = \{x \in X : \| \mid x \mid \wedge u \| < \varepsilon\}.$$

The collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with un-convergence. This topology is said **un-topology** which is a Hausdorff linear topology. Since each $V_{\varepsilon,u}$ is solid, then this topology is locally solid .

- Y. Deng, M. O'Brien, V.G. Troitsky, Unbounded norm convergence in Banach lattices, Positivity, 21 (2017) 963-974.
- M. Kandic, M.A.A. Marabeh, V.G. Troitsky, Unbounded norm topology in Banach lattices, J. Math. Anal. Appl. 451 (2017) 259-279.

Quotient un-topology

Let X be a Banach lattice with un-topology and I be a normed closed ideal of X. If $\mathfrak{B} = \{V_{\varepsilon,u} : 0 \neq u \in X^+, \varepsilon > 0\}$ is a base of zero neighborhoods for un-topology on X, then

$$\begin{aligned}\mathfrak{B}_{\frac{X}{I}} &= \{\pi(V_{\varepsilon,u})\ ;\ V_{\varepsilon,u} \in \mathfrak{B}\} \\ &= \{x + I\ ;\ x \in V_{\varepsilon,u}\} \\ &= \{x + I\ ;\ \||x| \wedge u\| < \varepsilon\},\end{aligned}$$

is a base of zero neighborhoods for un-topology on $\frac{X}{I}$.

Theorem

Let I be a normed closed ideal of a Banach lattice X. Then $\dot{V}_{\varepsilon,u} \subseteq V_{\varepsilon,\dot{u}}$. Moreover, if $\|u\| < \varepsilon$, then $V_{\varepsilon,\dot{u}} = \dot{V}_{\varepsilon,u}$.

Note: The first part of Theorem shows that the quotient map is un-continuous.

Example

Let $X = \mathbb{R}^2$. By coordinatewise order and Euclidean norm, X is a Banach lattice.
Consider $I = \{(x, 0) : \ x \in \mathbb{R}\}$ which is a normed closed ideal of $\mathbb{R}^2$.
Consider $u = (1, 0)$ and $\varepsilon = \frac{1}{2}$. Note that $\|u\| = 1$ and $\|u\| \nleq \varepsilon = \frac{1}{2}$. We have

$$\begin{aligned}
\dot{V}_{\varepsilon,u} &= \{\dot{x} : \ x \in V_{\varepsilon,u}\} \\
&= \{(a,b) + I : \ \| \mid (a,b) \mid \wedge (1,0)\| < \frac{1}{2}\} \\
&= \{(a,b) + I : \ \|(\mid a \mid \wedge 1, |b| \wedge 0)\| < \frac{1}{2}\} \\
&= \{(a,b) + I : \ |a| < \frac{1}{2}\} \subsetneqq \frac{\mathbb{R}^2}{I}.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
V_{\varepsilon,\dot{u}} &= \{\dot{x} :\ \| \mid \dot{x} \mid \wedge \dot{u} \| < \frac{1}{2}\} \\
&= \{(a,b)+I :\ \| \mid (a,b)+I \mid \wedge ((1,0)+I) \| < \frac{1}{2}\} \\
&= \{(a,b)+I :\ \| \mid (a,b)+I \mid \wedge I \| < \frac{1}{2}\} \\
&= \{(a,b)+I :\ \| I \| < \frac{1}{2}\} \quad \text{(note that } \|I\| = 0) \\
&= \{(a,b)+I :\ a,b \in \mathbb{R}\} = \frac{\mathbb{R}^2}{I}.
\end{aligned}
$$

Therefore $\dot{V}_{\varepsilon,u} \neq V_{\varepsilon,\dot{u}}$.

Proposition

Let X be a Banach lattice, I be a normed closed ideal of X and (x_α) be a net in X. Then

(a) If $x_\alpha \xrightarrow{un} x$ in X, then $\dot{x}_\alpha \xrightarrow{un} \dot{x}$ in $\frac{X}{I}$.

(b) If $x_\alpha \xrightarrow{w} x$ in X, then $\dot{x}_\alpha \xrightarrow{w} \dot{x}$ in $\frac{X}{I}$.

(c) If $x_\alpha \xrightarrow{uaw} x$ in X, then $\dot{x}_\alpha \xrightarrow{uaw} \dot{x}$ in $\frac{X}{I}$.

(d) If $x_\alpha \xrightarrow{u} x$ in X, then $\dot{x}_\alpha \xrightarrow{u} \dot{x}$ in $\frac{X}{I}$.

Example

Consider the Banach lattice ℓ_∞ and its normed closed ideal c_0. Consider $u = (1, 1, 1, \dots)$ and the sequence $(x_n) \subseteq \ell^\infty$ which $x_n = (0, 0, ..., 0, 1, 1, ...)$ with n zeros at the head for all $n \in \mathbb{N}$. This is a decreasing sequence in ℓ_∞ with infimum 0, and then it converges to zero in order in ℓ_∞. On the other hand, $x_n - u \in c_0$ for all $n \in \mathbb{N}$. This yields that $\pi(x_n) = \pi(u)$ is a (non-zero) constant sequence in the quotient Banach lattice $\frac{\ell^\infty}{c_0}$ which does not converge to $\dot{0}$ in order.

Result: The quotient mapping $\pi : X \to \frac{X}{I}$ is not generally order continuous.

Example

Since (x_n) is order convergent, then it is unbounded order convergent.

On the other hand, $\pi(x_n)$ is order bounded and we know that for order bounded sequences, order convergence and unbounded order convergence are equivalent.

It follows that $\pi(x_n)$ is not unbounded order continuous.

Result: The quotient map is not uo-continuous, in generally.

N. Gao, D. H. Leung, F. Xanthos, Duality for unbounded order convergence and applications, Springer International Publishing AG (2017).

Proposition: Let X be a Banach lattice and I be a normed closed ideal in X. Let $x \in X$ be an atom in X such that $x \notin I$. Then $x + I$ is an atom in $\frac{X}{I}$.
Remark: If $x \in I$, then $x + I = I$ is the zero element of quotient space which is not an atom in $\frac{X}{I}$, even x is an atom in X.
Theorem: If X is an atomic Banach lattice and I is a normed closed ideal of X, then $\frac{X}{I}$ is an atomic Banach lattice.

Example

Suppose that Banach lattice $X = \mathbb{R}^2$ and $I = \{(x, 0) : \ x \in \mathbb{R}\}$ and $J = \{(0, y) : \ y \in \mathbb{R}\}$ as normed closed ideals in $\mathbb{R}^2$. The elements $(a, 0)$ and $(0, a)$ are atoms in X for all $a > 0$.
So X is an atomic Banach lattice ($X = \mathbb{R}^2$ generated by (1,0) , (0,1)). With considering I, since $(a, 0) \in I$, then $(a, 0) + I$ is the zero of quotient space $\frac{X}{I}$. Therefore $(a, 0) + I$ is not an atom in $\frac{X}{I}$ but $(0, a) + I$ is an atom in $\frac{X}{I}$. Similarly, $(0, a) + J$ is not an atom in $\frac{X}{J}$ but $(a, 0) + J$ is an atom in $\frac{X}{J}$.

Thank you for your attentions.

Questions?