Orthogonally Additive polynomials : A New Frontier in Riesz Space Theory

ELMILOUD CHIL
University of Tunis Elmanar

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In this lecture, we focus on the systematic study of orthogonally additive polynomials as a new and promising area within Riesz space theory. A central challenge in this domain is understanding how these polynomials interact with other mathematical structures, particularly Riesz spaces.

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Despite their potential, the structural properties of orthogonally additive polynomials especially within the context of Riesz spaces have remained elusive until recently. Their study is significant not only from an algebraic standpoint but also within the framework of infinite-dimensional analysis, notably in the theory of holomorphic functions on infinite-dimensional spaces.

Introduction

One of the relevent problems in Operator Theory is to describe orthogonally additive polynomials via linear operators. This problem can be treated in different manner, depending on domains and codomains on which polynomials act. Interest in orthogonally additive polynomials on Banach lattices originates in the work of **Sundaresan**, where the space of n-homogeneous orthogonally additive polynomials on the Banach lattices l_p and $L_p\,[0,1]$ was characterized. It is only recently that the class of such mappings have been getting more attention. We are thinking here about works on orthogonally additive polynomials and holomorphic functions and orthosymmetric multilinear mappings on different Banach lattices and also $\mathbb{C}^*$-algebras. Proofs of the aforementioned results are strongly based on the representation of this spaces as vector spaces of extended continuous functions. So they are not applicable to general Riesz spaces. That is why we need to develop new approaches. Actually, the innovation of this work consist in making a relationship between orthogonally additive homogeneous polynomilas and orthosymmetric multilinear mappings which leads to a constructively proofs of **Sundaresan** results.

Historical

- **1991 : Sundaresan** On ℓ_p and $L_p[0,1]$

$$P(f) = \int f^n g d\mu.$$

- **2005 : Garcia , Villaneva, Carando, Lassale, Zalduendo** : On $C(X)$

$$P(f) = \int_X f^n g d\mu.$$

- **2006 : Benyamini, Lassale, LIanova** : On Banach lattices
- **2008 : Plazuelos, Peralta, Villanueva** : On C^*-algebras
- **2010-2012 : Carando, Lassale, Zalduendo, Jaramillo, Prieto, Zalduendo :** On holomorphic functions on open subsets of $C(X)$
- **2012 Bu, Buskes :** Banach lattices (Tensor products).
- **2012 Chil, Meyer** : On Uniformluy complete Riesz spaces
- **2015 Chil, Mokaddem :** On Riesz spaces
- **2020 Chil, Dorai** : On Riesz spaces by using Topological approach
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Unlocking the Secrets of Orthogonally Additive Polynomials in Riesz space Theory

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- A multilinear mapping $T : E^n \longrightarrow F$ is said to be orthosymmetric if $T(x_1, ..., x_n) = 0$ whenever $x_1, ..., x_n \in E$ satisfy $x_i \perp x_j$ for some $i \neq j$.
- Let E be a vector lattice and let F be a topological space. A map $P : E \to F$ is called a homogeneous polynomial of degree n (or a n-homogeneous polynomial) if $P(x) = \psi(x, .., x)$, where ψ is a n-multilinear map from E^n into F.
- A homogeneous polynomial, of degree n, $P : E \to F$ is said to be orthogonally additive if $P(x+y) = P(x) + P(y)$ where $x, y \in E$ are orthogonally (i.e. $|x| \wedge |y| = 0$).
- We denote by $\mathcal{P}_0(^nE, F)$ the set of n-homogeneous orthogonally additive polynomials from E to F.

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orthosymmetric multilinear mapping

- A bilinear map $T : E \times E \longrightarrow F$ is positive if $T(x,y) \geq 0$ whenever $(x,y) \in E^+ \times E^+$, and is order bounded if given $(x,y) \in E^+ \times E^+$ there exists $a \in F^+$ such that $|T(z,w)| \leq a$ for all $(0,0) \leq (z,w) \leq (x,y) \in E \times E$
- $T : E \times E \longrightarrow F$ is $(r.u)$ continuous if $x_n, y_n \longrightarrow 0$ $(r.u)$ in E implies that $T(x_n, y_n) \longrightarrow 0$ in F.

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- The set $\mathcal{L}_b(E)$ of all order bounded operators on E is an ordered vector space with respect to the pointwise operations and order. The positive cone of $\mathcal{L}_b(E)$ is the subset of all positive operators.
- An element T in $\mathcal{L}_b(E)$ is referred to as an orthomorphism if, for all $x, y \in E$, $|T(x)| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$. Under the ordering and operations inherited from $\mathcal{L}_b(E)$, the set $Orth(E)$ of all orthomorphisms on E is an Archimedean Riesz space.
- The Riesz algebra E is said to be an f-algebra whenever $x \wedge y = 0$ then $xz \wedge y = zx \wedge y = 0$ for all $z \in E^+$.
- If E is a Riesz space then the Riesz space $Orth(E)$ is an f-algebra with respect to the composition as multiplication. Moreover the identity map on E is the multiplicative unit of $Orth(E)$. In particular, the f-algebra $Orth(E)$ is semiprime and commutative.
- If E is an f-algebra with unit element, then the mapping $\pi : x \longrightarrow \pi_x$ from E into $Orth(E)$ is a Riesz and algebra isomorphism, where $\pi_x(y) = xy$ for all $y \in E$.

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- A Dedekind complete Riesz space E is said to be *universally complete* whenever every set of pairwise disjoint positive elements has a supremum.
- Every Archimedean Riesz space E has a unique (up to a Riesz isomorphism) universally completion denoted E^u, ie., there exists a unique universally complete Riesz space such that E can be identified with an order dense Riesz subspace of E^u.
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Theorem

Let E be a Riesz space, F be a Hausdorff t.v.s. (not necessarily a Riesz spaces) and let $\varphi : E \times E \to F$ be a $(r.u)$ continuous orthosymmetric bilinear map then φ is symmetric

Theorem

Let E be a Riesz space, F be a Hausdorff t.v.s., and let $T : E^n \to F$ be a $(r.u)$ continuous orthosymmetric multilinear map. If $\sigma \in S(n)$ is a permutation then

$$T(x_1, ..., x_n) = T(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

for all $x_1, ..., x_n \in E$.

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$$T(\pi_1(x_1), ..., \pi_n(x_n)) = T(x_1, ..., \pi_1 ... \pi_n(x_n))$$

for all $x_1, ..., x_n \in E$ and $\pi_1, ..., \pi_n \in Orth(E)$.

orthogonally additive homogeneous polynomials : Representations theorems

Let E be an Archimedean vector lattice, F be a Hausdorff topological vector space (not necessarily a vector lattice), $\psi : E^n \longrightarrow F$ be a $(r.u)$ continuous orthosymmetric multilinear map. Then there exists a linear operator $T_P : \overset{n}{\underset{i=1}{\pi}} E^{ru} \longrightarrow F$ such that

$$\psi(x_1, .., x_n) = T_P(x_1..x_n).$$

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We denote by $\mathcal{P}_{C0}(^nE, F)$ the set of continuous n-homogeneous orthogonally additive polynomials from E to F.

Let E be an Archimedean vector lattice, F be a Hausdorff topological vector space (not necessarily a vector lattice) and let $P \in \mathcal{P}_{C0}(^{n}E, F)$.

- Then ψ (its associated symetric multilinear map) is orthosymmetric.
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Our approach fails for the non continuous case

Structure Problem

Let E be the Riesz space of all real valued functions f on $[0,1]$ satisfying that there is a finite subset $(x_i)_{1\leq i\leq n}$ such that $0 = x_0 < x_1 < .. < x_n = 1$ and on each interval $[x_{i-1}, x_i)$ $f(x) = m_i(f)x + b_i(f)$ and $T(f,g) = m_0(f)b_0(g)$.

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- $P \in \mathcal{P}_{co}({}^nE, F)$ with E unital f-algebra $\Leftrightarrow$
- $P((x_1^n + .. + x_m^n)^{\frac{1}{n}}) = P(x_1) + .. + P(x_m) \Leftrightarrow$
- $P((x_1^n + x_2^n)^{\frac{1}{n}}) = P(x_1) + P(x_2) \Leftrightarrow$
- $P((x_1..x_n)^{\frac{1}{n}}) = \psi_P(x_1, .., x_n)$

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Polymorphisms and Polyorthomorphisms

- $P \in \mathcal{P}_{ob}({}^nE, F)$ with F Dedekind complete $|P|(x) = |T_p|(x^n)$ for all $x \in E^+$
- $\mathcal{P}_{ob}({}^nE, F^\delta)$ is a Riesz space.
- F Dedekind complete then $\mathcal{P}_{ob}({}^nE, F) \sim \mathcal{L}_b(\overset{n}{\underset{i=1}{\pi}} E^{ru}, F)$
- $P \in \mathcal{P}_{ob}({}^nE)$ is a polyorthomorphisms ($P \in \mathcal{P}_{Orth}({}^nE)$) if $T_p : \overset{n}{\underset{i=1}{\pi}} E^{ru} \to E^u$ is a weak orthomorphisms . So $P \in \mathcal{P}_{ob}({}^nE)$ and $x \perp y \Rightarrow Px \perp y$.
- $P \in \mathcal{P}_{ob}({}^nE, F)$ is a Polymorphisms ($P \in \mathcal{P}_{Hom}({}^nE)$) if $T_p : \overset{n}{\underset{i=1}{\pi}} E^{ru} \to F$ is a Riesz homomorphisms. so $P \in \mathcal{P}_{ob}({}^nE)$ and $x \perp y \Rightarrow Px \perp Py$

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Nakano type-Theorem

The null ideal of an arbitrary $P \in \mathcal{P}_{0b}(^nE, F)$ is the set

$$N_P = \{x \in E : |P|(|x|) = 0\} = \{x \in E : |T_P|(|x^n|) = 0\}$$

and its carrier is the band

$$C_P = N_P^d.$$

$N_P = N_{|P|}$

- Let $P \in (\mathcal{P}_{ob}(^nE, F))^+$ and $0 \leq x \leq y$ then $0 \leq P(x) \leq P(y)$
- N_P is an ideal of E and its a band whenever P is order continuous

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Nakano type theorem for order continuous n-homogeneous orthogonally additive polynomials :

Let $P, Q \in \mathcal{P}_{ob}({}^nE, \mathbb{R})^+$ such that P or Q is order continuous. Then the following are equivalents

$P \perp Q \Leftrightarrow C_P \subset N_Q \Leftrightarrow C_Q \subset N_P \Leftrightarrow C_P \perp C_Q$

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Structure Problem

Let $P \in \mathcal{P}_{ob}(^nE, C(Y))$ where $C(Y)$ is the space of continuous functions in a topological Hausdorff space Y. Then

$$N_P = \bigcap_{y\in Y} N_{\delta_y \circ P} \text{ and } C_P = \Big(\sum_{y\in Y} C_{\delta_y \circ P}\Big)^{dd}$$

where δ_y is the evaluation at y.

Structure Problem

Theorem

Let $P, Q \in \mathcal{P}_{ob}(^nE, C(Y))^+$ such that P or Q is order continuous. Then following

1. $C_P \perp C_Q$
2. $P \perp Q$
3. $\delta_y \circ P \perp \delta_y \circ Q$ *for all* $y \in Y$.
4. $C_{\delta_y \circ P} \perp C_{\delta_y \circ Q}$ *for all* $y \in Y$.

satisfies $1 \Rightarrow 2 \Rightarrow 3 \Leftrightarrow 4$.

Nakano theorem fails in the case of orthogonally additive polynomials between general Riesz spaces

Example

Consider the two positive order continuous orthogonally additive 2-homogeneous polynomials $P, Q : L_2[0,1] \to L_2[0,1]$ defined by

$$P(f) = (\int_0^1 f(x)^2 dx)\mathcal{X}_{[0,\frac{1}{2}]} \text{ and } Q(f) = (\int_0^1 f(x)^2 dx)\mathcal{X}_{[\frac{1}{2},1]}.$$

Observe that $P \perp Q$ because for every $0 \leq f \in L_2[0,1]$ we have $0 \leq (P \wedge Q)(f) \leq P(f) \wedge Q(f) = 0$. Thus $P \wedge Q = 0$ holds in $\mathcal{P}_{oc}({}^2L_2[0,1])$. However

$$N_S = N_T = \{0\} \text{ and so } C_T = C_S = L_2[0,1]$$

proving that C_T and C_S are not disjoints sets. On the other hand it is clear that for all $y \in [0,1]$ we have

$$\delta_y \circ P = 0 \text{ or } \delta_y \circ Q = 0$$

Theorem

Let $P, Q \in \mathcal{P}_{oc}(^nE, C(Y))^+$ *Then the following are equivalents :*

1. $C_P \perp C_Q$
2. $C_{\delta_y \circ P} \perp C_{\delta_z \circ Q}$ *for all* $y, z \in Y$.

The vector space $\mathcal{P}_{oc}({}^nE, \mathbb{R})$ will be called the *order continuous n-homogeneous orthogonally additive polynomial* ***dual*** *of E* and will be denoted

$$\mathcal{P}_{oc}({}^nE, \mathbb{R}) = ({}^nE)^{\sim_p}.$$

Since $\mathbb{R}$ is a Dedekind complete Riesz space, it follows that $({}^nE)^{\sim_p}$ is precisely the vector space generated by the positive orthogonally additive n-homogeneous polynomials. Moreover $({}^nE)^{\sim_p}$ is a Dedekind complete Riesz space. In general, there is no guarantee that a Riesz space supports any non trivial orthogonally additive polynomials of degree greater than one. Thus the order polynomial dual of E may happen to be the trivial space. As an example the Riesz space $({}^nL_p[0,1])^{\sim_p}$ is isometrically isomorphie to $L_q[0,1]$ for all $n < p$. When $n > p$ there are no non zero n-homogeneous orthogonally additive polynomials on $L_p[0,1]$. Therefore $({}^nL_p[0,1])^{\sim_p} = \{0\}$ for all $n > p$.

Now from the fact that $({}^nE)^{\sim_p}$ is again a Riesz space. Thus we can consider the order dual of $({}^nE)^{\sim_p}$ which is $(({}^nE)^{\sim_p})^{\sim} = \mathcal{L}_b(({}^nE)^{\sim_p}, \mathbb{R})$, the space of order bounded linear functional on $({}^nE)^{\sim_p}$. For each $x \in E$ an order bounded linear functional $\widehat{x}$ can be defined on $({}^nE)^{\sim_p}$ via the formula

$$\widehat{x}(P) = P(x) \text{ for all } P \in ({}^nE)^{\sim_p}.$$

Clearly, $x \geq 0$ implies $\widehat{x} \geq 0$. Also, since $P_\alpha \downarrow 0$ in $({}^nE)^{\sim_p}$ holds if and only if $P_\alpha(x) \downarrow 0$ in $\mathbb{R}$ for all $x \in E^+$. Then $\widehat{x}$ is order continuous linear functional on $({}^nE)^{\sim_p}$. Thus a positive map $x \to \widehat{x}$ can be defined from E to $((({}^nE)^{\sim_p})^{\sim})_n$, the space of order continuous linear functional on $({}^nE)^{\sim_p}$. This map is called the canonical embedding of E into $((({}^nE)^{\sim_p})^{\sim})_n$ which is one to one when $({}^nE)^{\sim_p}$ separates the points of E, that is for all $x \neq y$ there exists $P \in ({}^nE)^{\sim_p}$ such that $P(x) \neq P(y)$.

1. Let E an Archimedean Riesz space and $x \in E$. Then

$$\widehat{x}^{+} = \begin{cases} \widehat{x} & \text{if} \quad n \text{ is even} \\ \widehat{x^{+}} & \text{if} \quad n \text{ is odd} \end{cases}$$

2. For every $x \in E$ and $\lambda \in \mathbb{R}$ we have $\widehat{\lambda x} = \lambda^n \widehat{x}$.
3. For every $x, y \in E$ such that $x \perp y$ we have $\widehat{x+y} = \widehat{x} + \widehat{y}$.
4. If $0 \leq x \leq y$ then $\widehat{0} = 0 \leq \widehat{x} \leq \widehat{y}$.
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Consider $P \in (({}^nE)^{\sim_p})^+$ and $x \in E$. According to we can see that

$$\begin{aligned}\widehat{x}^+(P) &= \sup\left\{Q(x) : Q \in \mathcal{P}_{ob}({}^nE, \mathbb{R})^+ \text{ such that } 0 \leq Q \leq P\right\} \\ &= \sup\left\{T_Q(x^n) : T_Q \in \mathcal{L}_b(\prod_{i=!}^{n} E^{ru}, \mathbb{R})^+ \text{ such that } 0 \leq T_Q \leq T_P\right\} \\ &= T_P((x^n)^+)\end{aligned}$$

First if n is even then $x^n \geq 0$ therefore $T_P((x^n)^+) = T_P(x^n) = P(x) = \widehat{x}(P)$. Secondly if n is odd writing

$$\begin{aligned}T_P((x^n)^+) &= T_P(((x^+ - x^-)^n)^+) = T_P(((x^+)^n - (x^-)^n))^+) \\ &= T_P((x^+)^n) = P(x^+) = \widehat{x^+}(P).\end{aligned}$$

Thus

$$\widehat{x}^+ = \begin{cases} \widehat{x} & \text{if } \quad n \text{ is even} \\ \widehat{x^+} & \text{if } \quad n \text{ is odd} \end{cases}$$

Now we are able to announce a Nakano type theorem for orthogonally additive homogeneous polynomials.

Theorem

Let E be an Archimedean Riesz space. Then the embedding map

$$\begin{array}{rccc} \wedge : & E & \longrightarrow & (((^nE)^{\sim_p})^{\sim})_n \\ & x & \longrightarrow & \widehat{x} \end{array}$$

is an order continuous orthogonally additive n-homogeneous polynomial whose range is order dense in $(((^nE)^{\sim_p})^{\sim})_n$, the order continuous dual of $\mathcal{P}_{oc}(^nE, \mathbb{R})$.

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We have already mentioned that the canonical embedding of E into $((^nE)^{\sim_p})^{\sim}$ is an order continuous orthogonally additive n-homogeneous polynomial. So we need only to prove that the set $\{\widehat{x}, x \in E\}$ is order dense in $(((^nE)^{\sim_p})^{\sim})_n$. To this end, let $0 < \phi \in (((^nE)^{\sim_p})^{\sim})_n$ so by the order continuity of ϕ and the fact that $\phi \neq 0$ there exists $0 < P \in C_\phi = N_\phi^d$. Now, from the fact that P is order continuous and $P \neq 0$ it follows that $C_P \neq \{0\}$. So pick $0 < x \in C_P$. if $\widehat{x} \wedge \phi = 0$ then by the classical Nakano theorem we have $\widehat{x}(C_\phi) = 0$. Consequently, $P(x) = 0$ which implies that $x \in N_P \cap C_P = \{0\}$ which is impossible. Then $\widehat{x} \wedge \phi > 0$. Let $0 < \phi_1 = \widehat{x} \wedge \phi \leq \widehat{x}$ holds in $((^nE)^{\sim_p})^{\sim}$ for some $x \in E$. Now from the fact that $((^nE)^{\sim_p})^{\sim}$ is an Archimedean Riesz space and the fact that $\widehat{x} \neq 0$ there exists let $0 < \epsilon < 1$ such that $\psi = \phi_1 \vee \epsilon\widehat{x} - \epsilon\widehat{x} = (\phi_1 - \varepsilon\widehat{x})^+ > 0$.

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By the order continuity of ψ there exists $0 < P \in C_\psi$. Now it follows from the order continuity of P that $C_P \neq \{0\}$. We claim that we can choice $0 < y \in C_P$ such that $z = y \wedge \epsilon x > 0$. Assume by way of contradictions that $\epsilon x \in C_P^d = N(P)$ which implies that $\widehat{x}(P) = P(x) = 0$. On the other hand $0 \leq \psi \leq \phi_1 = \widehat{x} \wedge \phi \leq \widehat{x}$ hence $0 \leq \psi(P) \leq \widehat{x}(P) = 0$. Thus $\psi(P) = 0$ therefore $P \in C_\psi \cap N_\psi = \{0\}$ then $P = 0$ which is impossible. So there exists $0 < y \in C_P$ such that $z = y \wedge \epsilon x > 0$. Now, We claim that z satisfies

$$0 < \widehat{z} \leq \phi_1$$

in $(({}^nE)^{\sim_p})^{\sim}$. First to see that $0 < \widehat{z}$ holds, note that $0 \leq z \leq y$ and $y \in C_P$ which is an ideal. So $0 < z \in C_P$ which implies that $P(z) \neq 0$ so $\widehat{z}(P) \neq 0$, thus $0 < \widehat{z}$. Now, we claim that $\widehat{z} \leq \phi_1$. To this end, assume by way of contradictions that $\theta = (\widehat{z} - \phi_1)^+ > 0$. By the order continuity of θ there exists $0 < Q \in C_\theta$. On the other hand by Lemma we have $\widehat{z} \leq \varepsilon\widehat{x}$ because $z \leq \varepsilon x$ which implies that

$$0 \leq \theta = (\widehat{z} - \phi_1)^+ \leq (\varepsilon\widehat{x} - \phi_1)^+ = (\phi_1 - \varepsilon\widehat{x})^-.$$

•

Which implies that $\theta \perp \psi$ so $C_\theta \perp C_\psi$. On the other hand $P \in C_\psi$, $Q \in C_\theta$ which implies by theorem that $P \perp Q$ again by the same theorem [] we get

$$Q(C_P) = \{0\}.$$

Therefore by applying lemma once more and from the fact that $z \leq y$ we have

$$0 < \theta(Q) = (\widehat{z} - \phi_1)^+(Q) \leq \widehat{z}(Q) \leq \widehat{y}(Q) = Q(y) = 0.$$

Thus $\theta(Q) = 0$ which implies that $Q \in N_\theta \cap C_\theta = \{0\}$ then $Q = 0$ which is impossible. Hence

$$0 < \widehat{z} \leq \phi_1 \leq \phi.$$

Consequently the set $\{\widehat{x}, x \in E\}$ is order dense in $(((^nE)^{\sim_p})^{\sim})_n$, and the proof of the theorem is complete.

Orthogonally additive Polynomials in Riesz spaces

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1. A mapping $P : E \to F$ is a polynomial of degree $\leq n$ ($P \in \mathcal{P}_n(E, F)$) if

$$P = P_1 + .. + P_n \text{ where } P_k \in \mathcal{P}(^k E, F)$$

2. $P \in \mathcal{P}_n(E, F)$ is said to be orthogonally additive ($P \in \mathcal{P}_{no}(E, F)$) if $x \perp y \Rightarrow P(x + y) = P(x) + P(y)$
3. $P \in \mathcal{P}_{no}(E, F)$ if and only if $P_k \in \mathcal{P}_o(^k E, F)$ for all $1 \leq k \leq n$
4. $P \in \mathcal{P}_n(E, F)$ is $(r.u)$ continuous if and only if P_k is $(r.u)$ continuous for all $1 \leq k \leq n$

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$$E = F = C([0,1])\ P(f) = f + \int_0^1 f^2 \neq T(f + f^2) \text{ for all } T \text{ linear}$$

2. E is unital f-algebra if $P \in \mathcal{P}_n(E,F)$ then there exists $Q \in \mathcal{P}_n(E,F)$ such that

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Holomorphic mapping in Riesz spaces

E, F two complex Riesz spaces $H : E \longrightarrow F$ is said to be holomorphic if

$$H(x) = \sum_{1}^{+\infty} P_n(x) \text{ where } P_n \in \mathcal{P}(^nE, F)$$

H is called orthogonally additive ($H \in \mathcal{H}_o(E, F)$) if

$$\text{if } x \perp y \Rightarrow H(x+y) = H(x) + H(y)$$

1. $H \in \mathcal{H}_o(E,F)$ if and only if $P_n \in \mathcal{P}_o({}^nE,F)$ for all n
2. H is $(r.u)$ continuous if and only if P_n is $(r.u)$ continuous for all n
3. $P \in \mathcal{H}_{co}(E,F)$ is **valuation**

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Overall this research represents an important step forward in our understanding of orthogonally additive polynomials and their applications it has the potential to open up new avenues for research and innovation in a wide range of fields

THANK YOU FOR YOUR ATTENTION