Free Banach f-algebras

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Outline

1. Introduction to (Banach) f-algebras and examples

2. Free Banach f-algebras
   - Definition
   - Abstract construction
   - Towards a concrete representation

3. The representation problem
   - The finite-dimensional case
   - A study of the norm
   - Other properties

f-algebras

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Definition

A vector lattice algebra X (that is, a vector lattice together with a real algebra structure for which the product of positive elements is positive) is an *f-algebra* if, for every $a, b \in X$ with $a \wedge b = 0$,

$$(ac) \wedge b = 0 = (ca) \wedge b \quad \text{for all } c \in X_+.$$

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4. D'', where D is a commuting set of bounded Hermitian operators on a Hilbert space.

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Free Banach lattices

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Let E be a Banach space. The *free Banach lattice generated by* E is a Banach lattice $\mathrm{FBL}[E]$ together with an isometric embedding $\delta_E \colon E \to \mathrm{FBL}[E]$ such that, for every Banach lattice X and every operator $T \colon E \to X$, there exists a unique lattice homomorphism $\hat{T} \colon \mathrm{FBL}[E] \to X$ with $\|\hat{T}\| = \|T\|$ making the following diagram commutative:

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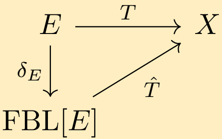

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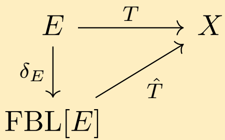

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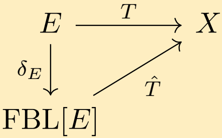

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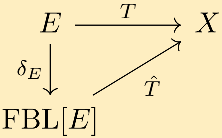

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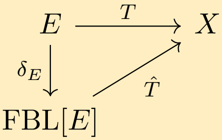

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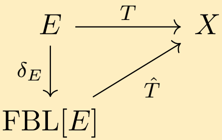

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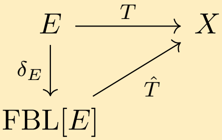

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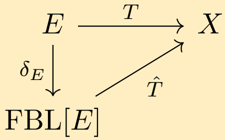

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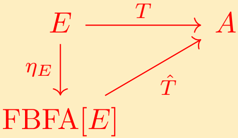

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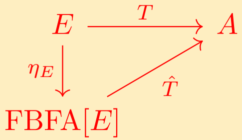

A general abstract argument (due to V. Troitsky and M. de Jeu) shows that this object exists. Can we construct it?

Abstract construction (I): FAFA

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A *lattice, linear and algebraic (LLA) expression* is a formal expression $\Phi[t_1, \ldots, t_n]$ involving finitely many variables, the linear and lattice operations, and a product. An LLA expression is said to *vanish* on a vector lattice algebra X if $\Phi(x_1, \ldots, x_n) = 0$ for every $x_1, \ldots, x_n \in X$.

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Theorem (M. Henriksen and J. R. Isbell)

Let Φ be an LLA expression. If Φ vanishes on $\mathbb{R}$, then it vanishes on every Archimedean f-algebra.

Abstract construction (II): FAFA

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Corollary

Let E be a vector space. The free Archimedean f-algebra generated by E is

$$\mathrm{FAFA}[E] = \mathrm{VLA}\{\delta_x \colon x \in E\} \subseteq \mathbb{R}^{E^\#},$$

where $\delta_x(\omega) = \omega(x)$ for every $\omega \in E^\#$, together with the linear map $\delta_E \colon E \to \mathrm{FAFA}[E]$.

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- Not all elements of $\mathrm{FAFA}[E]$ are positively homogeneous functions! (Compare with $\mathrm{FVL}[E]$.)

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- Hence, even if E is normed, we cannot represent $\mathrm{FAFA}[E]$ inside $C(B_{E^*})$ (because there exist non-zero elements of $\mathrm{FAFA}[E]$ that vanish on B_{E^*}).

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- Keep this fact in mind; I will surprise you later.

Abstract construction (III): FNFA and FBFA

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Construction of FNFA and FBFA

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Who is $\ker\rho \subseteq \mathrm{FAFA}[E]$? Can we give a nice description of $\mathrm{FNFA}[E]$?

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Who is $\ker\rho \subseteq \mathrm{FAFA}[E]$? Can we give a nice description of $\mathrm{FNFA}[E]$? Yes, we can. But first we need a new tool.

A new tool to compute $\ker \rho$

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Theorem (Structure theorem)

For every Banach f-algebra A there exist a Banach lattice X, a compact Hausdorff space K and a contractive injective lattice-algebra homomorphism $R\colon A \to X_0 \oplus_\infty C(K)$.

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Let Φ be an LLA expression. If Φ vanishes on $[-1, 1]$, then it also vanishes on the unit ball of every normed f-algebra.

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Proposition

Let E be a Banach space. Let ρ be the greatest submultiplicative lattice seminorm on $\mathrm{FAFA}[E]$ such that $\rho(\delta_x) \le \|x\|$ for all $x \in E$. Then

$$\ker \rho = \{\, f \in \mathrm{FAFA}[E] : f|_{B_{E^*}} = 0 \,\}.$$

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- Remember: $\mathrm{FAFA}[E]$ does not embed in $C(B_{E^*})$, i.e., the restriction map

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- So R induces an injective lattice-algebra homomorphism $R\colon \mathrm{FNFA}[E] \to C(B_{E^*})$ which must be contractive.

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  is not injective.
- Its kernel is

$$\ker R = \{ f \in \mathrm{FAFA}[E] : f|_{B_{E^*}} = 0 \}.$$

  Wait...This is precisely $\ker \rho$!
- So R induces an injective lattice-algebra homomorphism $R\colon \mathrm{FNFA}[E] \to C(B_{E^*})$ which must be contractive.
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- That's nice...But what about $\mathrm{FBFA}[E]$? Equivalently, what about the norm in $\mathrm{FNFA}[E]$?
- We seriously doubt that a simple, explicit expression for the free norm exists. Still, many things can be said about it. We present them in relation with the "representation problem."

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Theorem

Let E be a Banach space. The inclusion map $\mathrm{FNFA}[E] \to C(B_{E^*})$ extends to an injective lattice-algebra homomorphism $\mathrm{FBFA}[E] \to C(B_{E^*})$ if and only if $\mathrm{FBFA}[E]$ is semiprime.

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To find out whether $\operatorname{FBFA}[E]$ is semiprime, we still need to be able to say something about the norm, at least for some E.

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Theorem

Let E be a finite-dimensional Banach space. The free Banach f-algebra $\mathrm{FBFA}[E]$ is lattice-algebra isomorphic to $C([0,1] \times S_{E^*})$ with the pointwise order, and product

$$(f \star g)(r,u) = r f(r,u) g(r,u).$$

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Definition

Let E be a Banach space. For every $f \in \mathrm{FBFA}[E]$, define $\tau_E(f)$ to be the least positive number such that, if A is a semiprime finite-dimensional Banach f-algebra, and $T\colon E \to A$ is contractive, then $\tau_E(f) \geq \|\hat{T}f\|$ for every $f \in \mathrm{FBFA}[E]$.

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Theorem

If E is a finite-dimensional Banach space, then $\|f\| = \tau_E(f)$ for every $f \in \mathrm{FBFA}[E]$.

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Proposition

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Proposition

If E is contractively complemented in a space with a monotone basis, then τ_E coincides with the free norm in $\mathrm{FBFA}[E]$.

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is given by $\bar{T}f = f \circ T^*$. This is useful to study the properties of $\bar{T}$. These are not as clean as in the case of the FBL (for instance, T bijective does not imply $\bar{T}$ injective!).

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