Uncomplemented $\ell^{p,\infty}$

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It is enough to extend maps $T : F \to \ell^p(n)$ with control of norms.

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Observe that Pisier for $L^{p,\infty}$ implies injectivity of $L^{p,\infty}(\mu)$ in the category of Weak L^p spaces with positive maps. That is, every lattice isomorphic copy of $L^{p,\infty}(\mu)$ in a Weak L^p space is positively complemented.

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2. $\forall\ T : F \to X$, where X has upper p-estimate, $\exists\ Y$ with upper p-estimate, $\exists j : X \to Y$ lattice isomorphic embedding and $S : E \to Y$ such that $S\iota$ extends jT.

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Works also for $FBL^{(p)}$ and L^p. It gives an alternative solution of the "subspace/embedding" for $FBL^{(p)}$ without using Piser.

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$\mathcal{S}$ is an unconditional sequence space. The unit vector basis is a weakly null sequence.

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Therefore, (y_i^*) is relatively weakly compact.

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The End

Thank You