Automatic continuity of certain operators on ordered Banach spaces

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Automatic continuity of operators enjoying additional properties plays significant role in various applications. For example, a nice theory exists on automatic continuity of algebra homomorphisms. Another classical case is the automatic continuity of positive operators from Banach lattices to normed lattices. We are going to discuss how far one can go for general classes of operators from Banach spaces to topological vector spaces. Below, all vector spaces are real and operators linear.

Apparently, the importance of continuity of operators from Banach space to a normed space lies in the Uniform Boundedness Principle for families of continuous operators.

Let $T : X \to Y$ be an operator from a BS X to a TVS (Y, τ). In absence of an additional structure in X not much can be said about conditions providing boundedness of T.

Usage of the first size (finite) of sets gives no proper subclass, because

Every operator $T : X \to Y$ takes finite subsets into bounded.

Whereas, the second size (compact) returns us to bounded operators:

T takes compact subsets into bounded $\Longleftrightarrow$ T is bounded.

Indeed, if T is not bounded, there exist an absorbing $U \in \tau(0)$ and a sequence (x_n) in B_X with $Tx_n \notin n^2 U$. So, the image $\left\{T(\frac{x_n}{n})\right\}_{n=1}^{\infty} \cup \{0\}$ of compact set $\{\frac{x_n}{n}\}_{n=1}^{\infty} \cup \{0\}$ is not bounded.

The case when X is a Hilbert space was studied by S. Gorokhova [*Filomat* (2024)]. A modification of proof of Lemma 1.2 in her paper gives:

An operator $T : \mathcal{H} \to Y$ from a Hilbert space $\mathcal{H}$ to a TVS Y is bounded $\iff$ T takes orthonormal sets into bounded sets.

Let us drop (for a while) additional structures on X and consider:

Question 1

- *Is an operator $T : X \to Y$ from a BS X to a TVS Y bounded whenever it is bounded on each normalized (unconditional) basic sequence of X?*

The conditional part of Question 1 is open even when X is reflexive, whereas the unconditional part has a negative answer due to result of W.T. Gowers and B. Maurey [The Unconditional Basic Sequence Problem, *JAMS* (1993)], where a BS X was constructed with no unconditional basic sequence. For such an X, every linear functional on X is bounded along bounded unconditional basic sequences simply because of absence of such sequences. So, take any unbounded $f : X \to \mathbb{R}$.

Let us consider the case of the vector lattice structure in domain.

In the recent paper [E.E., N. Erkurșun-Özcan, S. Gorokhova, d-Operators in Banach Lattices: arXiv:2401.08792] devoted to operators possessing some properties along disjoint bounded sequences, no essential progress was achieved in the question on boundedness of operators bounded along such sequences.

More precisely, the Banach lattice version of Question 1 is still open.

Question 2

- *Is an operator from a BL to a TVS bounded, whenever it is bounded on each normalized disjoint sequence?*

Even the following extreme version of Question 2 seems to be open.

Question 3

- *Let f be a linear functional on a BL such that f is zero on each disjoint normalized sequence. Is f bounded?*

In what follows, we focus on the case of ordered Banach spaces in the domain. As main results below relay on the Baire category theorem, they have generalizations for completely metrizable ordered TVSs. In the present talk such a general case will not be discussed.

Where it is suitable, we formulate corresponding results in a slightly more general setting of collectively qualified sets of operators.

We need the following convergences in ordered vector spaces.

A net (x_α) in an OVS X

- *order converges* to x (o-converges to x, or $x_\alpha \xrightarrow{o} x$) if there exists a net $g_\beta \downarrow 0$ in X such that, for each β there is α_β satisfying $x_\alpha - x \leq \pm g_\beta$ for $\alpha \geq \alpha_\beta$.
- *relative uniform converges* to x (ru-converges to x, or $x_\alpha \xrightarrow{ru} x$) if, for some $u \in X_+$ there exists an increasing sequence (α_n) of indices with $x_\alpha - x \leq \pm\frac{1}{n}u$ for $\alpha \geq \alpha_n$.

An ONS X has *order continuous* norm if $X \ni x_\alpha \xrightarrow{\text{o}} 0 \Longrightarrow \|x_\alpha\| \to 0$. The following proposition extends a BL-result (which seems to be well known) to the OBS-setting.

Proposition 1

(Lemma 2.5; E.E.: arXiv:2503.18834)

Let X be an OBS with closed normal cone and order continuous norm. Then, $x_\alpha \xrightarrow{\text{o}} 0 \Longleftrightarrow x_\alpha \xrightarrow{\text{ru}} 0$ in X.

We shall also use the following collectively qualified sets of operators. The individual definitions come immediately by taking sets of one operator. Let $\mathcal{T}$ be a set of operators from X to Y.

Definition 1

When X and Y are OVSs, $\mathcal{T}$ is

a) *collectively order bounded* ($\mathcal{T} \in \mathbf{L}_{ob}(X, Y)$) *if, for every order interval* $[a, b]$ *in* X *there exists* $[u, v] \subseteq Y$ *with* $\mathcal{T}[a, b] \subseteq [u, v]$.

b) *collectively order continuous* ($\mathcal{T} \in \mathbf{L}_{oc}(X, Y)$) *if, for every* $x_\alpha \xrightarrow{\text{o}} 0$ *in* X *there exists a net* $g_\beta \downarrow 0$ *in* Y *such that, for each* β *there is* α_β *with* $Tx_\alpha \leq \pm g_\beta$ *for all* $\alpha \geq \alpha_\beta$ *and* $T \in \mathcal{T}$.

c) *collectively* ru*-continuous* (*or,* $\mathcal{T} \in \mathbf{L}_{ruc}(X, Y)$) *if, for every* $x_\alpha \xrightarrow{\text{ru}} 0$ *in* X *there exist an* $u \in Y_+$ *and an increasing sequence* (α_n) *of indices with* $\pm Tx_\alpha \leq \frac{1}{n}u$ *for all* $\alpha \geq \alpha_n$ *and* $T \in \mathcal{T}$.

Definition 2

When X is an OVS and (Y, τ) is a TVS, $\mathcal{T}$ is

a) *collectively order-to-topology bounded* ($\mathcal{T} \in \mathbf{L}_{o\tau b}(X, Y)$) *if* $\mathcal{T}[a, b]$ *is* τ*-bounded for every* $[a, b] \subseteq X$. *If, additionally* Y *is a NS, we write* $\mathcal{T} \in \mathbf{L}_{onb}(X, Y)$.

b) *collectively order-to-topology continuous* ($\mathcal{T} \in \mathbf{L}_{o\tau c}(X, Y)$) *if, for every* $x_\alpha \xrightarrow{\text{o}} 0$ *in* X *and* $U \in \tau(0)$, *there exists* α_U *with* $Tx_\alpha \in U$ *for all* $\alpha \geq \alpha_U$ *and* $T \in \mathcal{T}$. *If, additionally* Y *is a NS, we write* $\mathcal{T} \in \mathbf{L}_{onc}(X, Y)$.

The definition of collectively ru-to-topology continuous $\mathcal{T}$ *is obtained via replacement of* $x_\alpha \xrightarrow{\text{o}} 0$ *by* $x_\alpha \xrightarrow{\text{ru}} 0$.

Let X and Y be OVSs. Then

$$r_1\mathcal{T}_1 + r_2\mathcal{T}_2, \mathcal{T}_1 \cup \mathcal{T}_2 \in \mathbf{L}_{ob}(X, Y)$$

for every $r_1, r_2 \in \mathbb{R}$ and nonempty subsets $\mathcal{T}_1$, $\mathcal{T}_2$ of $\mathbf{L}_{ob}(X, Y)$. The same is true for $\mathbf{L}_{oc}(X, Y)$, $\mathbf{L}_{ruc}(X, Y)$, etc.

For other ways of constructing collectively qualified sets of operators, first consider a VL X and $x \in X$. Then, $\{y^\sim : |y| \le |x|\} \in \mathbf{L}_{oc}(X^\sim, \mathbb{R})$ by [Prop.2.3; E.E.: arXiv:2408.03671].

Next, let us consider one more construction.

Proposition 2

(Prop.2.4; E.E.:arXiv:2408.03671)

Let $\mathcal{T}$ a set of positive operators from an Archimedean VL X to a Dedekind complete VL Y. Then,

$$\mathcal{T}^{\delta} \in \mathbf{L}_{oc}(X^{\delta}, Y) \Longleftrightarrow \mathcal{T} \in \mathbf{L}_{oc}(X, Y),$$

where $\mathcal{T}^{\delta} = \{T^{\delta} : T \in \mathcal{T}\}$, and $T^{\delta} : X^{\delta} \to Y$ is unique linear extension of additive map $X^{\delta}_{+} \ni y \to \sup\limits_{x \in [0,y] \cap X} Tx$ to the Dedekind completion X^{δ}.

Proof.

Regularity of X in X^δ ensures $\mathcal{T}^\delta \in \mathbf{L}_{oc}(X^\delta, Y) \Longrightarrow \mathcal{T} \in \mathbf{L}_{oc}(X, Y)$.

Let $\mathcal{T} \in \mathbf{L}_{oc}(X, Y)$ and $x_\alpha \xrightarrow{o} 0$ in X^δ. Then, there is $y_\beta \downarrow 0$ in X^δ such that, for each β there exists α_β with $|x_\alpha| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. Since X is majorizing in X^δ, we may assume $(y_\beta) \subset X$.
As $\mathcal{T} \in \mathbf{L}_{oc}(X, Y)$ and $y_\beta \downarrow 0$ in X, there is a net $g_\gamma \downarrow 0$ in Y such that, for each γ, there is β_γ with $|Ty_\beta| \leq g_\gamma$ for all $\beta \geq \beta_\gamma$ and $T \in \mathcal{T}$. Then,

$$|T^\delta x_\alpha| \leq T^\delta |x_\alpha| \leq T^\delta y_{\beta_\gamma} = Ty_{\beta_\gamma} \leq g_\gamma$$

for all $\alpha \geq \alpha_{\beta_\gamma}$ and $T \in \mathcal{T}$, and hence $\mathcal{T}^\delta \in \mathbf{L}_{oc}(X^\delta, Y)$. □

The following is a collective OVS-version of order boundedness of order continuous operators from an Archimedean VL to a VL.

Theorem 1

(Thm.2.1; E.E., N.Erkursun-Özcan, S.Gorokhova.: Res.Math. (2025))
If X is an Archimedean OVS with a generating cone and Y is an OVS, then $\mathbf{L}_{oc}(X,Y) \subseteq \mathbf{L}_{ob}(X,Y)$.

Collectively order-to-topology bounded sets quite often agree with collectively ru-to-topology continuous ones.

Theorem 2

(Thm.2.1; E.E.: arXiv:2505.02200)

If X is an OVS and (Y, τ) a TVS, then $\mathbf{L}_{o\tau b}(X, Y) \subseteq \mathbf{L}_{r\tau c}(X, Y)$.
If additionally X_+ is generating, then $\mathbf{L}_{o\tau b}(X, Y) = \mathbf{L}_{r\tau c}(X, Y)$.

Proof.

If, in contrary, $\mathcal{T} \in \mathbf{L}_{o\tau b}(X,Y) \setminus \mathbf{L}_{r\tau c}(X,Y)$ then, for some $x_\alpha \xrightarrow{\text{ru}} 0$ there exists an absorbing $U \in \tau(0)$ such that, for every α there exist $\alpha' \geq \alpha$ and $T_\alpha \in \mathcal{T}$ with $T_\alpha x_{\alpha'} \notin U$. As $x_\alpha \xrightarrow{\text{ru}} 0$, for some $u \in X_+$ there is an increasing sequence (α_n) satisfying $\pm n x_\alpha \leq u$ for all $\alpha \geq \alpha_n$. In view of $\mathcal{T} \in \mathbf{L}_{o\tau b}(X,Y)$, $\mathcal{T}[-u,u] \subseteq NU$ for some $N \in \mathbb{N}$. Since $nx_\alpha \in [-u,u]$ for $\alpha \geq \alpha_n$ and $[\alpha_n]' \geq \alpha_n$, then $nx_{[\alpha_n]'} \in [-u,u]$. So, for every n, we have $T_{\alpha_n}(nx_{[\alpha_n]'}) \in NU$. In particular, $T_{\alpha_N}(x_{[\alpha_N]'}) \in U$ which is absurd.

Now, suppose X_+ is generating, and let $\mathcal{T} \in \mathbf{L}_{r\tau c}(X,Y) \setminus \mathbf{L}_{o\tau b}(X,Y)$. Since $X = X_+ - X_+$, there exist $x \in X_+$ and an absorbing $U \in \tau(0)$ with $\mathcal{T}[-x,x] \not\subseteq nU$ for every $n \in \mathbb{N}$. Find sequences (x_n) in $[-x,x]$ and (T_n) in $\mathcal{T}$ with $T_n x_n \notin nU$ for all n. Since $\frac{1}{n}x_n \xrightarrow{\text{ru}} 0$, there exists a sequence (n_k) such that $T(\frac{1}{n}x_n) \in U$ for all $n \geq n_k$ and $T \in \mathcal{T}$. In particular, $T_{n_1}x_{n_1} \in n_1 U$, a contradiction. □

The next corollary of Theorem 4 may be viewed as a topological version of [Thm.2.8; E.E., N.E-Ö, S.G.: Res.Math. (2025)].

Corollary 1

(Thm.2.3; E.E.: arXiv:2505.02200)

If X is an Archimedean OVS with a generating cone and (Y, τ) a TVS. Then $\mathbf{L}_{o\tau c}(X, Y) \subseteq \mathbf{L}_{o\tau b}(X, Y)$.

We list further consequences of Theorem 4 (see, [E.E.: arXiv:2505.02200] for more details).

Corollary 2

Let Y be a TVS. The following holds.

i) *Every order-to-topology bounded operator from an OVS X to Y is ru-to-topology continuous.*

If, in addition X_+ is generating, then order-to-topology bounded operators from X to Y agree with ru-to-topology continuous.

ii) *Every order-to-topology continuous operator from an Archimedean OVS X with a generating cone to Y is order-to-topology bounded.*

In particular, for every operator T from a normed lattice to a TVS:

$$\boxed{\text{T is ru-to-topology continuous} \iff \text{T is order-to-topology bounded}}$$

The following result generalizes [Thm.2.1; E.E.: arXiv:2408.03671] and [Thm.2.4; E.E., N.E-Ö, S.G.: Res.Math. (2025)]

Theorem 3

(Thm.2.5; E.E.: arXiv:2505.02200)

Let X be an OBS with a closed generating cone and (Y, τ) *a TVS. Then* $\mathbf{L}_{o\tau b}(X, Y) \subseteq \mathbf{L}_{n\tau b}(X, Y)$.

We include short proof of Theorem 3 based on the Krein – Smulian theorem.

Proof.

Let $\mathcal{T} \in \mathbf{L}_{o\tau b}(X, Y)$. Suppose, in contrary, $\mathcal{T} \notin \mathbf{L}_{n\tau b}(X, Y)$. By the Krein – Smulian theorem, $\alpha B_X \subseteq B_X \cap X_+ - B_X \cap X_+$ for some $\alpha > 0$, and hence $\mathcal{T}(B_X \cap X_+)$ is not τ-bounded. Then, there exists an absorbing $U \in \tau(0)$ with $\mathcal{T}(B_X \cap X_+) \not\subseteq nU$ for every $n \in \mathbb{N}$. So, for some sequences (x_n) in $B_X \cap X_+$ and (T_n) in $\mathcal{T}$ we have $T_n x_n \notin n^3 U$ for all n. Set $x := \|\cdot\|\text{-}\sum_{n=1}^{\infty} n^{-2} x_n \in X_+$. Since $\mathcal{T} \in \mathbf{L}_{o\tau b}(X, Y)$ then $\mathcal{T}[0, x] \subseteq NU$ for some $N \in \mathbb{N}$. It follows from $n^{-2} x_n \in [0, x]$ that $T_n(n^{-2} x_n) \in NU \subseteq nU$ for large enough n. This is absurd, because $T_n(n^{-2} x_n) \notin nU$ for all n. □

The following corollary of Theorem 3 provides a kind of the Uniform Boundedness Principle for families of operators which are not assumed to be continuous a priory.

Corollary 3

(Thm.2.8; E.E., N.E-Ö, S.G.: Res.Math. (2025))

Let X be an OBS with a closed generating cone and Y be a NS. Then every collectively order-to-norm bounded set of operators from X to Y is uniformly bounded.

Corollary 4

i) *Every order-to-norm bounded operator from an OBS with a closed generating cone to a NS is bounded.*

ii) *Every order bounded operator from an OBS with a closed generating cone to an ONS with a normal cone is bounded.*

iii) *Every order-to-norm bounded operator from a BL to a NS is bounded.*

iv) *Every order bounded operator from a BL to a normed lattice is bounded.*

v) *Every positive operator from a BL to a normed lattice is bounded.*

As, bounded operators in normed spaces are continuous,

Corollary 5

(Prop.1.5; E.E.: arXiv:2503.18834)

Every order-to norm bounded operator from an OBS with a closed generating cone to a normed space is continuous.

In Banach spaces weak compact sets are bounded. So,

Corollary 6

Let X be an OBS with a closed generating cone and Y a BS, and $T : X \to Y$ maps order intervals into weak compact sets. Then T is continuous.

One more consequence of Theorem 3 provides the following automatic continuity result.

Corollary 7

Let X be an OBS with a closed generating cone and Y an ONS with a normal cone. Then every order bounded $T : X \to Y$ is continuous.

Collective boundedness of a semigroup generated by a single operator is known as power boundedness. Similarly, we say that an operator T on an ONS X is order-to-norm power bounded if $\{T^n\}_{n=0}^{\infty} \in \mathbf{L}_{onb}(X)$. We have two more consequences of Theorem 3.

Corollary 8

Let X be an OBS with a closed generating cone. Then every order-to-norm power bounded operator $T : X \to X$ is power bounded.

Corollary 9

Every order-to-norm bounded operator semigroup on an OBS with a closed generating cone is uniformly bounded.

A set $\mathcal{T}$ of operators from an OVS X to a TVS Y is

- *collectively* ru-*to-topology continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{ru\tau}(X, Y)$) if $\mathcal{T}x_\alpha \xrightarrow{\text{c-}\tau} 0$ whenever $x_\alpha \xrightarrow{\text{ru}} 0$.

Quite often the collectively norm-to-topology bounded sets agree with collectively ru-to-topology continuous ones.

Theorem 4

Let X be an OVS and (Y, τ) a TVS. Then $\mathbf{L}_{o\tau b}(X, Y) \subseteq \mathbf{L}_{ru\tau}(X, Y)$. If additionally X_+ is generating then $\mathbf{L}_{o\tau b}(X, Y) = \mathbf{L}_{ru\tau}(X, Y)$.

Proof. Assume, in contrary, $\mathcal{T} \in \mathbf{L}_{o\tau b}(X, Y) \setminus \mathbf{L}_{ru\tau}(X, Y)$. Then, $\mathcal{T}x_\alpha \overset{\text{c-}\tau}{\not\longrightarrow} 0$ for some $x_\alpha \xrightarrow{\text{ru}} 0$. So, there exists an absorbing $U \in \tau(0)$ such that, for every α there exist $\alpha' \geq \alpha$ and $T_\alpha \in \mathcal{T}$ with $T_\alpha x_{\alpha'} \notin U$.

Since $x_\alpha \xrightarrow{\text{ru}} 0$, for some $u \in X_+$ there exists an increasing sequence (α_n) of indices with $\pm n x_\alpha \leq u$ for $\alpha \geq \alpha_n$. It follows from $\mathcal{T} \in \mathbf{L}_{o\tau b}(X, Y)$ that $\mathcal{T}[-u, u] \subseteq NU$ for some $N \in \mathbb{N}$. Since $nx_\alpha \in [-u, u]$ for $\alpha \geq \alpha_n$ and $(\alpha_n)' \geq \alpha_n$ then $nx_{(\alpha_n)'} \in [-u, u]$, and hence $T_{\alpha_n}(nx_{(\alpha_n)'}) \in NU$ for every n. In particular, $T_{\alpha_N}(x_{(\alpha_N)'}) \in U$ which is absurd. We conclude $\mathbf{L}_{o\tau b}(X, Y) \subseteq \mathbf{L}_{ru\tau}(X, Y)$.

Now, suppose X_+ is generating, and let $\mathcal{T} \in \mathbf{L}_{ru\tau}(X,Y)$. Assume, in contrary, $\mathcal{T} \notin \mathbf{L}_{o\tau b}(X,Y)$. Since $X = X_+ - X_+$, there exist $x \in X_+$ and an absorbing $U \in \tau(0)$ with $\mathcal{T}[-x,x] \nsubseteq nU$ for every $n \in \mathbb{N}$.

Find sequences (x_n) in $[-x,x]$ and (T_n) in $\mathcal{T}$ with $T_n x_n \notin nU$ for all n. Since $\frac{1}{n}x_n \xrightarrow{\text{ru}} 0$ then $\mathcal{T}(\frac{1}{n}x_n) \xrightarrow{\text{c-}\tau} 0$. So, there exists a sequence (n_k) such that $T(\frac{1}{n}x_n) \in U$ for all $n \geq n_k$ and $T \in \mathcal{T}$. Then $T_{n_1}x_{n_1} \in n_1 U$, which is a contradiction. Therefore, $\mathcal{T} \in \mathbf{L}_{o\tau b}(X,Y)$.

The proof is complete.

Since $\mathbf{L}_b(X,Y) \subseteq \mathbf{L}_{onb}(X,Y)$ whenever X is a normal ONS and Y is a NS, the next result follows from Theorems 3 and 4.

Theorem 5

Let X be an OBS with a closed generating normal cone and Y a NS. Then $\mathbf{L}_{run}(X,Y) = \mathbf{L}_{onb}(X,Y) = \mathbf{L}_b(X,Y)$.

Corollary 10

Let T be an operator from an OBS with a closed generating normal cone to a NS. Then

$$T \text{ is ru-to-norm continuous} \iff$$

$$T \text{ is order-to-norm bounded} \iff T \text{ is bounded}.$$

Accordingly to [Thm.2.4; E.E., N.E-Özcan, S.G.: Res.Math. (2025)], $\mathbf{L}_{ruc}(X, Y) \subseteq \mathbf{L}_{ob}(X, Y)$, whenever X and Y are OVSs and X_+ is generating. Therefore, we have one more consequence of Theorem 3 that provides a kind of the Uniform Boundedness Principle.

Corollary 11

Let X be an OBS with a closed generating cone and Y an ONS with a normal cone. Then $\mathbf{L}_{ruc}(X, Y) \subseteq \mathbf{L}_{b}(X, Y)$.

In particular,

Corollary 12

Every ru-continuous operator from an OBS with a closed generating cone to an ONS with a normal cone is continuous.

Proposition 3

(Prop.2.11; E.E., N.E-Ö, S.G.: Res.Math. (2025))
Let X and Y be OBSs with closed generating cones, and $X^{\sim} \neq \emptyset$. Then every $T \in \mathbf{L}_{ob}(X, Y)$ is uniformly bounded iff Y_+ is normal.

Proof.

If Y_+ is not normal then the interval $[0, y_0]$ is not bounded for some $y_0 \in X_+$. Pick $f_0 \neq 0$ in $X^{\sim}$ and take $x_0 \in X$ with $f_0(x_0) = 1$. Then, $\{f_0 \otimes y\}_{y \in [0,y_0]} \in \mathbf{L}_{ob}(X, Y)$. However, the set $\{f_0 \otimes y\}_{y \in [0,y_0]}$ is not uniformly bounded since $\bigcup_{y \in [0,y_0]} (f_0 \otimes y)(x_0) = [0, y_0]$ is not bounded.

If Y_+ is normal then $\mathbf{L}_{ob}(X, Y) \subseteq \mathbf{L}_{onb}(X, Y)$ (Prop.2.6 in [E.E., N.E-Ö, S.G.: Res.Math. (2025)]). The rest follows from Theorem 5. □

Since $\mathbf{L}_{ruc}(X, Y) = \mathbf{L}_{ob}(X, Y)$ when X and Y have generating cones [Thm.2.4; E.E., N.E-Ö, S.G.: Res.Math. (2025)], we have the following consequence of Proposition 3 (which provides a converse to Corollary 11).

Corollary 13

Let X and Y be OBSs with closed generating cones, and $X^\sim \neq \emptyset$. Then every collectively ru-continuous set of operators from X to Y is bounded iff Y_+ is normal.

It is unknown, whether in Proposition 3 one can replace collectively order bounded sets by order bounded operators. More precisely,

Question 4

Let X and Y be OBSs with closed generating cones, $X^{\sim} \neq \emptyset$, and each order bounded operator from X to Y is bounded. Is Y_+ normal?

It is well known that all lattice norms that a make a vector lattice a Banach lattice are equivalent. This is an immediate consequence of continuity of every positive operator from a BL to a NL.

Let us consider the question whether one can obtain equivalentness of certain norms on an OVS by using automatic continuity of order bounded operators from an OBS with a closed generating cone to an ONS with a normal cone (it is provided by Corollary 7).

It is natural to consider norms which makes a cone of an OVS closed and normal. More precisely, we have the following.

Proposition 4

Let X be an OVS with a generating positive cone X_+. Then, all norms on X that make X a BS so that X_+ is closed and normal are equivalent.

Proof.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two complete norms on X under which X_+ is closed and normal. The identity operator $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is order bounded, and hence continuous by Corollary 7. By the same reason, its inverse is also continuous. This guarantees that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

We conclude the talk with the following proposition that is a partial case of [Lemma 2.1; E.E.: arXiv:2408.03671].

Proposition 5

Let X be a normal OBS and Y a NS. Then $\mathrm{L}_{owc}(X,Y) \subseteq \mathrm{L}_{onb}(X,Y)$.

Proof.

Let $T : X \to Y$ be ow-continuous but $T[0,u]$ is not bounded for some $u \in X_+$. As X is normal, $[0,u]$ is bounded, say $\sup\limits_{x\in[0,u]} \|x\| \le M$. Pick a sequence (u_n) in $[0,u]$ with $\|Tu_n\| \ge n2^n$, and set $y_n := \|\cdot\|\text{-}\sum\limits_{k=n}^{\infty} 2^{-k}u_k$ for $n \in \mathbb{N}$. Then $y_n \downarrow\geq 0$. Let $0 \le y_0 \le y_n$ for all $n \in \mathbb{N}$. Since $0 \le y_0 \le 2^{1-n}u$ and $\|2^{1-n}u\| \le M2^{1-n} \to 0$ then $y_0 = 0$ by [Thm.2.23; Aliprantis–Tourky: Cones and Duality]. Thus, $y_n \downarrow 0$. It follows from $T \in \mathrm{L}_{owc}(X,Y)$ that (Ty_n) is w-null, and hence is norm bounded, that is absurd because $\|Ty_{n+1} - Ty_n\| = \|T(2^{-n}u_n)\| \ge n$ for all $n \in \mathbb{N}$. □

Thank You for Your Attention!