1. Definition of O- and uO-convergence.

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net and x a point in a lattice L.

(i) $(x_\gamma)_{\gamma\in\Gamma}$ is said to *order converge* (O_2*-converge*) to $x \in L$ if there exists a directed set $M \subseteq L$ and a filtered set $N \subseteq L$ satisfying $\bigvee M = \bigwedge N = x$, and such that for every $(m,n) \in M \times N$, $(x_\gamma)_{\gamma\in\Gamma}$ is eventually contained in $[m,n]$. In this case we write $x_\gamma \xrightarrow{O_2} x$.

(ii) $(x_\gamma)_{\gamma\in\Gamma}$ is said to *unbounded order converge* (uO_2*-converge*) to $x \in L$, if

$$f_{s,t}(x_\gamma) \xrightarrow{O_2} f_{s,t}(x)$$

for every $s \leq t$, where $f_{s,t}(x) := (x \wedge t) \vee s$.

2. For lattices there is another definition for order convergence: $(x_\gamma)_{\gamma\in\Gamma}$ is said to O_1*-converge* to $x \in L$ if there exists an increasing net $(a_\gamma)_{\gamma\in\Gamma}$ and a decreasing net $(b_\gamma)_{\gamma\in\Gamma}$ such that $a_\gamma \uparrow x$, $b_\gamma \downarrow x$ and $a_\gamma \leq x_\gamma \leq b_\gamma$ (eventually).

Clearly, O_1-convergence implies O_2-convergence.

3. [ACW, 2023]

Let $F \subseteq L^L$. The net $(x_\gamma)_{\gamma\in\Gamma}$ is said to *FO_i-converge to x in L* if $(f(x_\gamma))_{\gamma\in\Gamma}$ is O_i-convergent to $f(x)$ for every $f \in F$.

Let L be a lattice and $F \subseteq L^L$. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in L that FO_2-converges to $x \in L$. Then $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that FO_1-converges to x. In particular:

- If a net $(x_\gamma)_{\gamma\in\Gamma}$ of L is O_2-convergent to $x \in L$, then $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that O_1-converges to x.
- If a net $(x_\gamma)_{\gamma\in\Gamma}$ of L is uO_2-convergent to $x \in L$, then $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that uO_1-converges to x.

4. The following assertions are easily verified.

   (i) If $x_\gamma \uparrow x$ in L, then $x_\gamma \xrightarrow{\mathrm{O}} x$. The dual statement for decreasing nets holds as well.

   (ii) If a net O-converges/uO-converges, then the limit is unique.

   (iii) If $(x_\gamma)_{\gamma\in\Gamma}$ is O-convergent to x, and eventually $(x_\gamma)_{\gamma\in\Gamma}$ is contained in $a^{\downarrow}$, then $x \le a$. The dual statement holds as well.

   (iv) If L is bounded, uO-convergence implies O-convergence.

   (v) If the lattice is distributive, the condition $s \le t$ in the definition of uO-convergence becomes redundant: $x_\gamma \xrightarrow{\mathrm{uO}} x$ iff $(x_\gamma \vee s) \wedge t \xrightarrow{\mathrm{O}} (x \vee s) \wedge t$ for every $s, t \in L$.

5. [Papangelou, 1964]

   When the lattice is a commutative ℓ-group (in particular, when it is a Riesz space) the unbounded-order convergence defined above coincides with the established notion of unbounded convergence on such structures.

6. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a lattice L.

   (i) If $x_\gamma \xrightarrow{\mathrm{uO}} x$, and eventually $(x_\gamma)_{\gamma\in\Gamma}$ is contained in $u^{\downarrow}$, then $x \le u$. The dual statement holds as well.

   (ii) If $(x_\gamma)_{\gamma\in\Gamma}$ is monotonic, the following implication holds:

$$x_\gamma \xrightarrow{\mathrm{uO}} x \quad \Longrightarrow \quad \begin{cases} \bigvee_\gamma x_\gamma = x \text{ (if the net is increasing)}, \\ \bigwedge_\gamma x_\gamma = x \text{ (if the net is decreasing)}. \end{cases}$$

7. **But:** The following example illustrates that the converse of (ii) above may fail, even in the context of distributive lattices. This stands in sharp contrast to the case of Riesz spaces, where uO-convergence is order continuous.

   Example 1 *Let L denote the collection of all the closed subsets of $\mathbb{R}$. When endowed with set inclusion, L forms a bounded distributive lattice. For $n \in \mathbb{N}$ let $X_n := [2^{-n}, \infty)$ and let $X := [0, \infty)$. Then $(X_n)_{n\in\mathbb{N}}$ is increasing and $\bigvee^L X_n = X$, i.e. $X_n \uparrow X$ in L. In particular, $X_n \xrightarrow{O} X$. On the other hand, if we let $A := (-\infty, -1]$ and $B := (-\infty, 0]$, then $(X_n \wedge B) \vee A = A$ for every $n \in \mathbb{N}$, but $(X \wedge B) \vee A = \{0\} \cup A$.*

8. Let L be a lattice. We recall that L satisfies the *join infinite distributive law (JID)* if for each $a \in L$ and $S \subseteq L$, whenever $\vee S$ exists, so does $\bigvee\{a \wedge s : s \in S\}$ and $a \wedge (\bigvee S) = \bigvee\{a \wedge s : s \in S\}$. Similarly, L satisfies the *meet infinite distributive law (MID)* if whenever $\bigwedge S$ exists, so does $\bigwedge\{a \vee s : s \in S\}$ and $a \vee (\bigwedge S) = \bigwedge\{a \vee s : s \in S\}$.

9. [AC 2025]

> A distributive lattice L is infinitely distributive if and only if $\mathfrak{u}O$-convergence is order continuous.

10. When L is an infinitely distributive lattice, an eventually order bounded net is uO-convergent to x iff it O-converges to x. In particular: In a bounded, infinitely distributive lattice, O-convergence and uO-convergence are the same.

11. For a subset X of a lattice L let

$$X_1^O := \{x \in L : \text{there exists a net in } X \text{ that O-converges to } x\},$$
$$X_1^{\mathfrak{u}O} := \{x \in L : \text{there exists a net in } X \text{ that uO-converges to } x\}.$$

For every ordinal number $\lambda > 0$ we can define the *λ-O-adherence* X_λ^O, and the *λ-$\mathfrak{u}O$-adherence* $X_\lambda^{\mathfrak{u}O}$ recursively: Set $X_0^O := X =: X_0^{\mathfrak{u}O}$ and

$$X_\lambda^O := \left(\bigcup_{\beta<\gamma} X_\beta^O\right)_1^O \qquad X_\lambda^{\mathfrak{u}O} := \left(\bigcup_{\beta<\lambda} X_\beta^{\mathfrak{u}O}\right)_1^{\mathfrak{u}O}.$$

12. The set X is said to be *O-closed* (resp. *uO-closed*) if $X = X_1^O$ (resp. $X = X_1^{\mathfrak{u}O}$). The set of all O-closed subsets of L forms a topology on L, called the *order topology*. The same can be said for the uO-closed sets and one can speak of the *uO-topology* as the topology given rise by the uO-closed subsets of L. The O-closure of $X \subseteq L$ is the smallest O-closed subset of L that contains X, i.e. the O-closure is the topological closure w.r.t. the order topology. Note that this will generally be larger than X_1^O. Similarly, the uO-closure is the smallest uO-closed subset of L that contains X.

13. For a lattice L:

    (i) The cuts $a^{\uparrow}$ and $a^{\downarrow}$ are O-closed and uO-closed.

    (ii) If X is a subset of an infinitely distributive lattice L, then $X_1^O \subseteq X_1^{uO}$.

    (iii) Let

$$\alpha := \min\{\lambda \geq 0 : X_\lambda^O = X_{\lambda+1}^O\},$$
$$\beta := \min\{\lambda \geq 0 : X_\lambda^{uO} = X_{\lambda+1}^{uO}\}.$$

    Then X_α^O coincides with the the O-closure (=topological closure w.r.t. the order topology) of X and X_β^{uO} with the uO-closure (= topological closure w.r.t. the uO-topology) of X. When L is infinitely distributive, we note that $X_\alpha^O \subseteq X_\beta^{uO}$.

14. [AC 2025]

    The O-closure and the uO-closure of a sublattice of an infinitely distributive lattice L coincide, and the resulting subset is again a sublattice of L.

15. In particular, a sublattice of an infinitely distributive lattice is O-closed iff it is uO-closed.

16. **But:** The condition of infinite distributivity is essential and cannot be replaced by the weaker assumption of distributivity.

    Example 2 *Consider the following subsets of* $2^{\mathbb{R}}$.

$$\begin{aligned} \mathcal{C}_- &:= \{(-\infty, a] : a \leq 0\} \\ \mathcal{C}'_- &:= \{(-\infty, a] : a < 0\} \\ \mathcal{C}_+ &:= \{[a, +\infty) : a \geq 0\} \\ \mathcal{C}'_+ &:= \{[a, +\infty) : a > 0\} \end{aligned}$$

    The ring L of subsets of $\mathbb{R}$ generated by $\mathcal{C}_- \cup \mathcal{C}_+$ consists of all subsets of $\mathbb{R}$ that have one of the following types: $\emptyset$, $(-\infty, -a]$, $[b, +\infty)$, $(-\infty, -a] \cup [b, +\infty)$, $\{0\}$, where $a, b \geq 0$. This forms a distributive lattice. The sub-ring Y generated by $\mathcal{C}'_- \cup \mathcal{C}'_+$ consists of all subsets of $\mathbb{R}$ that have one of the following types: $\emptyset$, $(-\infty, -a]$, $[b, +\infty)$, $(-\infty, -a] \cup [b, +\infty)$, where $a, b > 0$. Y is a sublattice of L. The O-closure $\overline{Y}$ of Y in L consists of the subsets of $\mathbb{R}$ that have one of following types: $\emptyset$, $(-\infty, -a]$, $[b, +\infty)$, $(-\infty, -a] \cup [b, +\infty)$, where $a, b \geq 0$. Observe that the infimum in $\overline{Y}$ of $(-\infty, 0]$ and $[0, +\infty)$ is equal to $\emptyset$, whereas the infimum taken in L equals $\{0\}$.

17. [AC 2025]

> Let L be an infinitely distributive lattice and $A \subseteq L$ be an ideal. Then $A_1^{O} = A_1^{uO}$ and both are uO-closed (and therefore O-closed) ideals.

18. In the above theorem, can ideals be replaced with regular sublattices. (Note that every ideal of a lattice is, in particular, a regular sublattice.) **Given an infinitely distributive lattice L and a regular sublattice, how many order/ unbounded-order adherences do we need to take to reach the O-closure of Y?**

19. [Gao&Leung 2018]

> Let L be an Archimedean Riesz space with the countable sup property and admitting a separating family of order-continuous positive linear functionals and let Y be a Riesz subspace of L. Then $Y_2^{O} = Y_1^{uO}$ covers the O-closure of Y.

20. Let P be a poset. For $D \subseteq P$ let $D^{\uparrow} := \{x \in P : x \geq d \ \forall d \in D\}$ and $D^{\downarrow} := \{x \in P : x \leq d \ \forall d \in D\}$. If $D = D^{\uparrow\downarrow}$, then we say that D is a lower-cut (l-cut) of P. The Dedekind-MacNeille completion of P, denoted by $\mathrm{DM}(P)$, is the set of all l-cuts of P, ordered with set inclusion. $\mathrm{DM}(P)$ forms a *complete lattice* satisfying the following properties.

    (a) $x^{\downarrow} \in \mathrm{DM}(P)$ for every $x \in P$ and the function $\varphi : P \to \mathrm{DM}(P) : x \mapsto x^{\downarrow}$ is isotone.

    (b) If $\{D_i : i \in I\} \subseteq \mathrm{DM}(P)$ then

    $$\bigvee_{i\in I}^{\mathrm{DM}(P)} D_i = \left(\bigcup_{i\in I} D_i\right)^{\uparrow\downarrow} \quad \text{and} \quad \bigwedge_{i\in I}^{\mathrm{DM}(P)} D_i = \bigcap_{i\in I} D_i\,.$$

    (c) $\varphi[P]$ is *join-dense* and *meet-dense* in $\mathrm{DM}(P)$, i.e.

    $$a = \bigvee^{\mathrm{DM}(P)} \{\varphi(x) \ : \ x \in P,\ \varphi(x) \leq a\}\,,$$

    and

    $$a = \bigwedge^{\mathrm{DM}(P)} \{\varphi(x) \ : \ x \in P,\ \varphi(x) \geq a\}\,,$$

    for every $a \in \mathrm{DM}(P)$. From this follows that φ preserves all suprema and infima that exist in P

    (d) Let $D \subseteq P$. Then

    $$D^{\downarrow} = \bigwedge^{\mathrm{DM}(P)} \varphi[D] = \bigvee^{\mathrm{DM}(P)} \varphi[D^{\downarrow}]\,,$$

    and

    $$D^{\uparrow\downarrow} = \bigvee^{\mathrm{DM}(P)} \varphi[D] = \bigwedge^{\mathrm{DM}(P)} \varphi[D^{\uparrow}]\,.$$

(e) The Dedekind-MacNeille completion of P is characterized – up to order-isomorphism – as the unique complete lattice containing P as a simultaneously join-dense and meet-dense sublattice.

21. **But:** There are distributive lattices that cannot be regularly (lattice) embedded in a complete and distributive (indeed, modular) lattice [Crawley 1961]. In particular, this means that the Dedekind-MacNeille completion of a distributive lattice need not be distributive.

Example 3 *The Dedekind-MacNeille completion of an infinitely distributive lattice need not be infinitely distributive. When endowed with the pointwise partial order,*

$$L := \{(0,b) : 0 \leq b < 1\} \cup \{(1,b) : 0 \leq b < +\infty,\, b \neq 1\}$$

forms an infinitely distributive lattice. It is easy to see that

$$\begin{aligned} DM(L) &= \{(0,b) : 0 \leq b < 1\} \cup \{(1,b) : 0 \leq b \leq +\infty\}, \\ L^\delta &= \{(0,b) : 0 < b < 1\} \cup \{(1,b) : 0 \leq b < +\infty\}. \end{aligned}$$

Let us show that L^δ (and hence $DM(L)$) does not satisfy the Join-Infinite Distributive Law. Let $x_n = (0, 1 - \frac{1}{n})$. Then $\bigvee^{L^\delta} x_n = (1,1)$ and $\left(\bigvee^{L^\delta} x_n\right) \wedge (1, \frac{1}{2}) = (1, \frac{1}{2})$. On the other hand, $\bigvee^{L^\delta} (x_n \wedge (1, \frac{1}{2})) = (0, \frac{1}{2})$.

22. If the poset P happens to be an Abelian and Archimedean ℓ-group, it is possible to endow

$$P^\delta := \mathrm{DM}(P) \setminus \{\emptyset, P\}$$

with a group structure to obtain a Dedekind complete ℓ-group containing the starting ℓ-group as a regular ℓ-subgroup [Clifford 1940].

23. The same happens if P is a Boolean lattice: If B is a Boolean lattice, then $\mathrm{DM}(B)$ is again a Boolean lattice [Stone-Glivenko].

24. [ABC –]

Let Y be a regular sublattice of an infinitely distributive lattice L. Assume that:

- $\mathrm{DM}(L)$ is infinitely distributive;
- $\mathrm{DM}(Y)$ (lattice) embeds regularly in $\mathrm{DM}(L)$.

Then $Y_1^O = Y_1^{uO} = \mathrm{DM}(Y) \cap L$. In particular, Y_1^O covers the O-closure of Y.

25. [Gao, Troitsky & Xanthos 2017]

If Y is a regular Riesz subspace of an Archimedean Riesz space L, then Y^δ (lattice) embeds regularly in L^δ.

26. In particular, we observe that:

If Y is a regular Riesz subspace of an Archimedean Riesz space L, then Y_1^O is O-closed.

27. What is the relationship between the Dedekind–MacNeille completion of a sublattice and that of its containing lattice? A positive result:

Let L be a lattice and $Y \subseteq L$ be a sublattice. Then

$$\mathfrak{i} : \mathrm{DM}(Y) \ni A \mapsto A^{\uparrow\downarrow} \in \mathrm{DM}(L)$$

is an order-embedding of $\mathrm{DM}(Y)$ into $\mathrm{DM}(L)$.

28. Definition:

Let Y be a sublattice of a lattice L.

- Y is said to have *Property (A)* if for $A \subseteq Y$, $x \in A^{\downarrow}$ and $y \in A^{\downarrow} \cap Y$, there exists $u \in A^{\downarrow} \cap Y$ such that $u \geq x \vee y$.
- Y is said to have *Property (B)* if for $A \subseteq Y$, $x \in A^{\uparrow}$ and $y \in A^{\uparrow} \cap Y$, there exists $u \in A^{\uparrow} \cap Y$ satisfying $u \leq x \wedge y$.

29. [AC 2025]

Let L be a lattice and $Y \subseteq L$ be a sublattice. Let $i : \mathrm{DM}(Y) \to \mathrm{DM}(L) : A \mapsto A^{\uparrow\downarrow}$ be the order-embedding described above.

- If Y satisfies Property (A), then i preserves arbitrary meets.
- If Y satisfies Property (B), then i preserves arbitrary joins.

30. If Y is a sublattice satisfying Properties (A) and (B), then $i[\mathrm{DM}(Y)]$ is a regular sublattice of $\mathrm{DM}(L)$.

31. [AC 2025]

Let L be an infinitely distributive lattice and $Y \subseteq L$ a sublattice satisfying Properties (A) and (B). Then $Y_1^O = Y_1^{uO} = \mathrm{DM}(Y) \cap L$. In particular, Y_1^O covers the O-closure of Y.

THANK YOU FOR YOU ATTENTION AND I HOPE TO SEE YOU IN MALTA FOR THE NEXT POSITIVITY CONFERENCE IN JUNE 2027.