Von Neumann-Maharam problem for vector lattices

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Von Neumann-Maharam problem

Let $(X, \mathcal{A}, \mu)$ be a measure space. Its *measure* algebra is $\mathcal{A}/\ker\mu$.

Von Neumann problem asks to characterize Boolean algebras which appear as measure algebras of finite measure spaces.

Alternatively, if A is a complete Boolean algebra, characterize when is there a countably disjointly additive functional $\mu : A \to [0, +\infty]$ (i.e. a *measure*) such that $\ker\mu = \{0_A\}$ (i.e. *strictly positive*), and which is *semi-finite* (if $\mu(a) = \infty$, there is $b \le a$ with $0 < \mu(b) < \infty$).

Von Neumann problem for vector lattices: characterization of those VL's F which can be order densely embedded into $L_0(\mu)$, where μ is a strictly positive semi-finite measure.

Note that the topology of (local) convergence in measure restricts to an order continuous Hausdorff topology on F.

Maharam problems are to characterize those VL's or BA's which admit such a topology and to ascertain whether having such a topology implies embeddability into $L_0(\mu)$ or admitting a measure.

Krein-Kakutani representation of a vector lattice

Everywhere F is an Archimedean vector lattice. F_e is an ideal in F generated by $e \in F$.

Theorem 1 (Krein-Kakutani Representation theorem)

F_e is isomorphic to a dense (linear) sublattice of $\mathcal{C}(K_e)$, for a compact Hausdorff K_e. We call K_e the ***Krein-Kakutani spectrum*** *of e.*

Note that K_e is the space of unital homomorphisms on $(F_e, |e|)$.

We will call a property of a vector lattice

- *Local* if F has it whenever F_e has it for every $e \in F_+$. (Countable) Dedekind completeness, completeness in σ-order convergence, relative uniform completeness, countable interpolation property...
- *Spectral* if F has it whenever $\mathcal{C}(K_e)$ has it for every $e \in F_+$. (Countable) projection property, countable supremum property, almost countable completeness, sufficiently many projections...

Every spectral property is local. All given examples of local properties are not spectral.

The countable supremum property

Let P be either a Boolean algebra or an Archimedean VL.

P has *the countable supremum property* (CSP) if for all $Q \subset P$ and $q \in P$ with $q = \bigvee Q$ there is a countable $Q' \subset Q$ so that $q = \bigvee Q'$, and *the countable chain condition* (CCC) if disjoint sets in P are countable.

A net $(p_\alpha) \subset P$ *σ-order converges* to $p \in P$ (denoted by $p_\alpha \xrightarrow{\sigma o} p$) if there is a countable $Q \subset P$ with $\bigwedge Q = 0_P$ so that for every $q \in Q$ there is α_q such that $|p_\alpha - p| \leq q$, for $\alpha \geq \alpha_q$. Clearly, $p_\alpha \xrightarrow{\sigma o} p \Rightarrow p_\alpha \xrightarrow{o} p$.

Proposition 1 (TFAE:)

- *P has the CSP;* • $\mathrm{o} = \sigma\mathrm{o}$; • *(If P is a BA:) P has the CCC;*
- *(If P is an AVL:) Order bounded disjoint sets in P are countable.*

$\mathcal{C}(K)$ has the CSP iff it has the CCC iff K has the CCC, i.e. every disjoint collection of opens subsets of K is countable.

If F is an AVL, then $F_{csp} := \{e \in F,\ F_e \text{ has the CSP}\}$ is the largest ideal in F with the CSP. Note that $e \in F_{csp}$ iff K_e has the CCC.

L_0 over a Boolean algebra

Let A be a complete Boolean algebra with the Stone space K_A, which is extremally disconnected.

Let $\mathcal{C}^\infty(K_A)$ be the set of all $f \in \mathcal{C}(K_A, [-\infty, +\infty])$ such that $f^{-1}(\pm\infty)$ is nowhere dense. Then $f + g$ is defined pointwise almost everywhere.

Let E be a vector lattice of Borel real-valued functions on K_A. Let $N = \{f \in E,\ K_A \backslash \ker f \text{ is meager}\}$, which is an ideal in E.

Proposition 2

$L_0(A) := \mathcal{C}^\infty(K_A) \simeq E/N$ as vector lattices. Moreover, there is a bijection between $L_0(A)$ and the collection of all σ-order continuous Boolean homomorphisms from $\{$Borel subsets of $\mathbb{R}\}$ into A.

A has the CCC iff $L_0(A)$ has the CCC iff it has the CSP.

A BA or AVL P is *weakly (σ, ∞)-distributive* if whenever $q_0 \geq Q_n \downarrow 0_P$, then $\bigwedge \{p \in P,\ \forall n \in \mathbb{N}\ \exists q_n \in Q_n \cap [0_P, p]\} = 0_P$.

A has this property iff $L_0(A)$ has it.

Maeda-Ogasawara-Vulikh representation of a VL

Recall that bands in F form a complete Boolean algebra $\mathcal{B}_F$.

A sublattice $E \subset F$ is *order dense* if $E \cap (0_F, f] \neq \varnothing$, for every $f > 0_F$. In this case $H \mapsto H \cap E$ defines an isomorphism from $\mathcal{B}_F$ onto $\mathcal{B}_E$.

Theorem 2 (Maeda-Ogasawara-Vulikh)

$L_0(\mathcal{B}_F)$ is the largest AVL containing F as an order dense sublattice.

We will call $K_F := K_{\mathcal{B}_F}$ the *Maeda-Ogasawara-Vulikh spectrum* of F, denote $F^u := L_0(\mathcal{B}_F)$ and call it the *universal completion* on F.

We will call a property is *horizontal* if F has it whenever F^u has it, and so it only depends on $\mathcal{B}_F$, or on K_F.

The CCC is horizontal (F has the CCC iff $\mathcal{B}_F$ has it) but not local.

Projection property is spectral but not horizontal. “Any sequence is contained in a principal ideal” property is neither horizontal nor local.

Weak (σ, ∞)-distributivity is both horizontal and spectral (F has it iff $\mathcal{B}_F$ has it iff for any $e \in F$ meager sets are nowhere dense in K_e).

Locally solid topologies

A *submeasure* is an order preserving $\rho : A \to \mathbb{R}$ with $\rho(0_A) = 0$ and $\rho(a \vee b) \leq \rho(a) + \rho(b)$, for any $a, b \in A$ (can be replaced either with $\rho(a \triangle b) \leq \rho(a) + \rho(b)$ or with disjoint subadditivity).

There is a (neither injective nor surjective) correspondence between submeasures on A and pseudo-norms on $L_0(A)$.

A group topology on A or F is *locally solid* if it has a base at 0 of solid sets. It is *order continuous (Lebesgue)* if $p_\alpha \xrightarrow{\mathrm{o}} p \Rightarrow p_\alpha \to p$ (and T_2).

Proposition 3

Locally solid topologies are generated by

- *For AVL's, by Riesz pseudo-norms, i.e. subadditive functionals whose balls are solid.*
- *For BA's, by submeasures.*

Moreover, a single Riesz pseudo-norm / submeasure is enough if the topology is first countable.

Lebesgue vector lattices and Boolean algebras

We say that A or F is *Lebesgue* if it admits a Lebesgue topology.

Proposition 4 (TFAE:)

- *F is Lebesgue;* • *F_e is Lebesgue, for every $e \in F$;*
- *F_{csp} is order dense and F_e is Lebesgue, for every $e \in F_{csp}$;*
- *$\mathcal{C}(K_e)$ is Lebesgue, for every $e \in F$;* • *$\mathcal{B}_F$ is Lebesgue;*
- *F embeds order densely into $\prod\limits_{i \in I} L_0(A_i, \mu_i)$, where each μ_i is a strictly positive order continuous sub-measure on a complete A_i.*

Hence, Lebesgue is both a spectral and horizontal property.

If A or F is Lebesgue then it is weakly (σ, ∞)-distributive.

Theorem 3 (Sarymsakov + Rubinstein + Chilin & Weber, 70s)

There is at most one Lebesgue topology on A. We denote it τ_A.

Dedekind complete $\Leftrightarrow$ τ_A-complete, and the CCC $\Leftrightarrow$ τ_A-metrizable.

Unbounded order convergence on vector lattices

Let F be an AVL. A net $(f_\alpha) \subset F$ *unbounded order* (uo) converges to $f \in F$ ($f_\alpha \xrightarrow{\mathrm{uo}} f$) if $|f_\alpha - f| \wedge h \xrightarrow{\mathrm{o}} 0_F$, for all $h \geq 0_F$.

Theorem 4 (B., Troitsky, 2022)

In $\mathcal{C}(X)$ we have $f_\alpha \xrightarrow{\mathrm{uo}} \mathbb{0}$ iff for every open $U \neq \varnothing$ and $\varepsilon > 0$ there is an open $\varnothing \neq V \subset U$ and α_0 such that $|f_\alpha|_{|V} \leq \varepsilon$, for $\alpha \geq \alpha_0$.

Similar criterions are also valid in $\mathcal{C}^\infty(X)$ and also in $L_p(\mu)$ with sets of positive measure instead of open sets, and also in $L_0(A)$.

Theorem 5 (Aliprantish, Burkinshaw, Conradie, Fremlin, Taylor,..)

A Hausdorff LS topology τ is weaker than uo iff τ is the weakest Lebesgue topology on F iff $\tau = \mathrm{u}\pi$, where π is arbitrary Lebesgue.

If F is Lebesgue, this topology exists and is unique; denote it by τ_F. It is metrizable iff F has the CCC. Topological completion of (F, τ_F) is F^u.

If $F = L_p(\mu)$: τ_F = local convergence in μ, and uo = a.e. for sequences.

Topological modification of a convergence

The *topological modification* $\mathrm{t}\eta$ of a convergence η is the "cotopology" formed by the η-closed sets.

Theorem 6 (Maharam, 1947)

A is Lebesgue and CCC iff $\mathrm{t}\sigma\mathrm{o}$ *is metrizable. In this case* $\tau_A = \mathrm{t}\sigma\mathrm{o} = \mathrm{to}$ *is generated by a strictly positive order continuous submeasure.*

Theorem 7 (Deng + de Jeu, 2024 & B., 2025)

If F is Lebesgue with the CSP, then $\tau_F = \mathrm{tuo}$.

$(f_n)_{n\in\mathbb{N}}$ *is* τ_F*-null iff each subsequence has a* uo*-null sub-subsequence.*
If F is atomless, the last condition implies CSP (if CH; false if MA+¬CH).

Question 1 (Open since 70s)

Is it always true that if A or F is Lebesgue then $\tau_A = \mathrm{to}$ *and* $\tau_F = \mathrm{tuo}$*?*

Note that if $e \in F_{csp}$, then F_e has the CCC, along with $\mathcal{B}_{F_e}$. Then, F_e is Lebesgue iff $\mathcal{B}_{F_e}$ is Lebesgue and has CCC.

Theorem 8 (Balcar, Fremlin, Główczyński, Jech, Pazák, Todorčević)

For a complete Boolean algebra A TFAE:

- *A is Lebesgue and has CCC;*
- *A has the CCC and $\mathrm{to} = \mathrm{t\sigma o}$ is Hausdorff;*
- *$\mathrm{t\sigma o}$ is regular;*
- *$\vee$ is $\mathrm{t\sigma o}$-continuous at $(0_A, 0_A)$;*
- *A is weakly (σ, ∞)-distr. and $\{0_A\}$ is a G_δ set with respect to $\mathrm{t\sigma o}$;*
- *A is weakly (σ, ∞)-distr. and $A = \bigcup\limits_{n\in\mathbb{N}} A_n$, where A_n's do not contain infinite disjoint sets;*
- *A has the CCC and **some** stronger version of weak distributivity.*

*(Complete + CCC + weakly (σ, ∞)-distr. $\Rightarrow$ Maharam) is **consistent**.*

*It is **also consistent** that there is a complete weakly (σ, ∞)-distributive CCC non-Maharam BA with a strictly positive Fatou submeasure.*

Question 2

Find a version of Theorem 8 for vector lattices. Remove completeness.

Total failure

Theorem 9 (Talagrand, 2005)

Maharam does NOT imply existence of a strictly positive measure.

A *charge* on A is a finitely disjointly additive functional.

Note that a charge is a measure iff it is σ-order continuous. Existence of a strictly positive charge yields the CCC. Hence, every strictly positive measure is order continuous.

Theorem 10 (Kantorovich + Vulikh + Pinsker, 1950 & Kelley, 1959)

There is a strictly positive finite measure on A iff there is a strictly positive charge on A and A is weakly (σ, ∞)-distributive.

Theorem 11 (Kelley, 1959 & Kalton + Roberts, 1983)

A locally solid topology on A is generated by charges iff it is uniformly exhaustive, i.e. for every neighborhood U of 0_A there is $n \in \mathbb{N}$ such that there are no disjoint n-tuples in $A \setminus U$.

Theorem 12 (Preliminary)

For an Archimedean vector lattice F TFAE:

- $\mathcal{B}_F$ *admits a strictly positive semi-finite measure* μ*;*
- F *embeds order densely into* $L_0(\mu)$*, where* μ *is as above;*
- F *is Lebesgue and* τ_F *is uniformly exhaustive;*
- F *is Lebesgue and* $\tau_F|_{[0_F,f]}$ *is locally convex, for every* $f \geq 0_F$*;*
- F_e *admits a locally convex Lebesgue topology, for every* $e \in F$*;*
- *For all* $e \in F$ *there is a non-zero order continuous functional on* F_e*;*
- $\mathcal{C}(K_e)^\delta$ *is a dual space, for every* $e \in F$*;*
- F *is weakly* (σ, ∞)*-distr. and* $\bigcup\limits_{\nu \in F_+^\sim} \ker(\nu \circ |\cdot|)^d$ *is order dense in* F*.*

This property is both spectral and horizontal.

THANK YOU!