Characterizing Riesz* homomorphisms via interval preserving order adjoints

Based on joint work with Anke Kalauch, Janko Stennder, and Onno van Gaans

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Characterizing Riesz homomorphisms

Theorem (Kim, Andô, 1975)

Let X and Y be Riesz spaces such that $Y^\sim$ is separating and let $T\colon X \to Y$ be a positive linear map. Then T is a Riesz homomorphism if and only if its order adjoint

$$T^\sim\colon Y^\sim \to X^\sim, \quad g \mapsto g \circ T,$$

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- Buskes, van Rooij, and van Haandel have introduced multiple generalizations of Riesz homomorphisms from Riesz spaces to ordered vector spaces.
- Is a similar characterization true for these notions of Riesz homomorphisms?

Preliminaries

Order adjoints and interval preserving operators

Let (X, K) and (Y, L) be ordered vector spaces. We denote by

$$X^{\sim} := \{f : X \to \mathbb{R};\, f \text{ is linear and order bounded}\}$$

the *order dual* of X. Let $T: X \to Y$ be linear and positive.

- The positive linear map

  $$T^{\sim}: Y^{\sim} \to X^{\sim}, \quad g \mapsto g \circ T$$

  is called the *order adjoint* of T.
- T is called *interval preserving* if $T[0, x] = [0, Tx]$ holds for all $x \in K$.

Pre-Riesz spaces

An ordered vector space (X, K) is called a *pre-Riesz space* if there exists a Riesz space Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y, i.e.,

$$\forall y \in Y: \quad y = \inf \{i(x);\, x \in X, i(x) \geq y\}\,.$$

The pair (Y, i) is called a *vector lattice cover* of X.

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The pair (Y, i) is called a *vector lattice cover* of X. If $i[X]$ generates Y as a Riesz space, i.e.,

$$\forall y \in Y\, \exists a_1, \ldots, a_n, b_1, \ldots, b_m \in X: \quad y = \bigvee_{j=1}^{n} i(a_j) - \bigvee_{k=1}^{m} i(b_k)$$

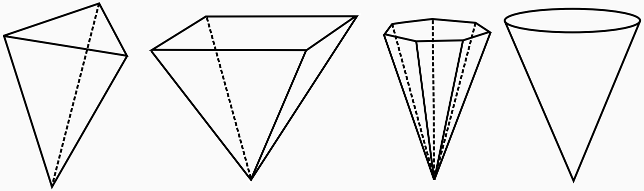
then (Y, i) is called the *Riesz completion* of (X, K). The Riesz completion is unique up to order isomorphism.

Typical examples for pre-Riesz spaces

Every vector lattice is a pre-Riesz space.

X Archimedean and directed $\implies$ X pre-Riesz $\implies$ X directed.

- Directed function spaces, e.g., $\mathrm{C}^n[0,1]$, $\mathrm{P}^n[0,1]$, $\mathrm{P}[0,1]$
- $\mathrm{L}^{\mathrm{r}}(X,Y)$ with X directed and Y Archimedean
- Finite-dimensional spaces with closed cones with non-empty interior



Notions of Riesz homomorphisms

Let (X, K) and (Y, L) be ordered vector spaces. A linear map $T\colon X \to Y$ is called a

- *Riesz* homomorphism* if

$$\forall \varnothing \neq F \subseteq X \text{ finite} : T\left[F^{\mathrm{u}\ell}\right] \subseteq T[F]^{\mathrm{u}\ell};$$

- *Riesz homomorphism* if

$$\forall x_1, x_2 \in X : \{Tx_1, Tx_2\}^{\mathrm{u}\ell} = T\left[\{x_1, x_2\}^{\mathrm{u}}\right]^{\ell}.$$

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$$T \text{ Riesz hom.} \overset{Y \text{ directed, Archimedean}}{\Longrightarrow} T \text{ Riesz* hom.} \implies T \text{ positive}$$

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If X and Y are Riesz spaces, then:

$$T \text{ Riesz hom.} \iff T \text{ Riesz* hom.}$$

and they coincide with the notion of Riesz homomorphisms between Riesz spaces.

The van Haandel extension

Theorem (Van Haandel, 1993)

Let X_1, X_2 be pre-Riesz spaces with respective vector lattice covers $(Y_1, i_1), (Y_2, i_2)$, and let $T\colon X_1 \to X_2$ be linear.

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1. *If there exists a Riesz homomorphism $T^\rho\colon Y_1 \to Y_2$ with $T^\rho \circ i_1 = i_2 \circ T$, then T is a Riesz* homomorphism.*

$$\begin{array}{ccc} X_1 & \xrightarrow{\;T\;} & X_2 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_2} \\ Y_1 & \xrightarrow{\;T^\rho\;} & Y_2. \end{array}$$

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1. *If there exists a Riesz homomorphism $T^\rho\colon Y_1 \to Y_2$ with $T^\rho \circ i_1 = i_2 \circ T$, then T is a Riesz* homomorphism.*
2. *If (Y_1, i_1) is the Riesz completion of X_1 and T is a Riesz* homomorphism, then there exists a unique Riesz homomorphism $T^\rho\colon Y_1 \to Y_2$ with $T^\rho \circ i_1 = i_2 \circ T$.*

$$\begin{array}{ccc} X_1 & \xrightarrow{T} & X_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ Y_1 & \xrightarrow{T^\rho} & Y_2. \end{array}$$

Riesz* homomorphisms characterized via order adjoints

The lattice-case

Theorem (Kim, Andô, 1975)

Let X and Y be Riesz spaces, $T\colon X \to Y$ linear and positive, and

$$T^{\sim}\colon Y^{\sim} \to X^{\sim}, \quad g \mapsto g \circ T,$$

its order adjoint.

1. *If $Y^{\sim}$ is separating[1] and $T^{\sim}$ is interval preserving, then T is a Riesz homomorphism.*
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[1] i.e., $\forall y \in Y : y = 0 \iff (\forall g \in Y^{\sim} : g(y) = 0)$

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Operators with interval preserving order adjoints

The proof for the classical result relies on the following result: If X is a Riesz space and $f: X \to \mathbb{R}$ is linear and positive, then

$$\forall x \in X : \quad f(x^+) = \max\{g(x);\ g \in [0, f]\}.$$

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Theorem (B., Kalauch, Stennder, van Gaans, 2025)

Let (X,K) be a directed ordered vector space and $f\colon X \to \mathbb{R}$ linear and positive. Then:

$$\forall x \in X: \quad \inf f\left[\{0,x\}^{\mathrm{u}}\right] = \max\{g(x);\ g \in [0,f]\}.$$

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With this new result, one can prove the following:

Theorem (B., Kalauch, Stennder, van Gaans, 2025)

Let (X, K) and (Y, L) be directed ordered vector spaces such that L^* determines positivity[2] *and let $T\colon X \to Y$ be linear and positive. If $T^\sim$ is interval preserving, then T is a Riesz homomorphism.*

[2] i.e., $\forall y \in Y : (\forall g \in L^* : g(y) \geq 0) \implies y \geq 0$

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Note: If Y is a Riesz space, then:

$$Y^\sim \text{ is separating } \iff L^* \text{ determines positivity.}$$

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Order adjoints of Riesz* homomorphisms

Observation: If (X, K) is a pre-Riesz space and (X^ρ, i_X) its Riesz completion, then i_X is a Riesz homomorphism.[4]

[4] i_X is even a complete Riesz homomorphism, i.e., $\inf A = 0 \Rightarrow \inf i_X[A] = 0$.

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Note: If X is a Riesz space, then:

$$(X^\rho, i_X) = (X, \mathrm{id}_X) \implies i_X^\sim = \mathrm{id}_{X^\sim} \implies i_X^\sim \text{ is interval preserving.}$$

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Summary

If (X, K) and (Y, L) are Riesz spaces:

$$T \text{ Riesz homomorphism} \underset{Y^{\sim} \text{ separating}}{\overset{}{\rightleftarrows}} T^{\sim} \text{ interval preserving}$$

Summary

If (X, K) and (Y, L) are Riesz spaces:

T Riesz homomorphism $\Longrightarrow$ $T^{\sim}$ interval preserving

T Riesz homomorphism $\Longleftarrow$ $T^{\sim}$ interval preserving ($Y^{\sim}$ separating)

If (X, K) and (Y, L) are pre-Riesz spaces and (X^{ρ}, i_X) is the Riesz completion of X:

T Riesz homomorphism $\Longleftarrow$ $T^{\sim}$ interval preserving (L^* det. pos.)

T Riesz homomorphism $\not\Uparrow\,\Downarrow$ T Riesz* homomorphism

T Riesz* homomorphism $\Longrightarrow$ $T^{\sim}$ interval preserving ($i_X^{\sim}$ interval preserving)

How strong is the condition that $i_X^{\sim}$ is interval preserving?

First counterexample: Namioka space

Consider $X = \{x \in \mathrm{C}[0,5];\ 2x(1) = x(0) + x(5)\}$. Then

$$T\colon X \to X, \quad x \mapsto (t \mapsto w(t)x(\alpha(t)))\,,$$

with

$$w(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ t-1 & \text{if } 1 < t \leq 3, \\ 5-t & \text{if } 3 < t \leq 5 \end{cases} \text{ and } \alpha(t) := \begin{cases} \frac{t}{3} & \text{if } 0 \leq t \leq 3, \\ 2t-5 & \text{if } 3 < t \leq 5 \end{cases}$$

is a (complete) Riesz homomorphism, but $T^{\sim}$ is not interval preserving.

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Idea: $\mathrm{ev}_0 \in [0, T^\sim \mathrm{ev}_3]$ and take any $x \in X$ with $x(0) < 0$ and $x(t) \ge 0$ for all $t \in [1,5]$. Then $Tx \ge 0$ and if there is $\varphi \in [0, \mathrm{ev}_3]$ with $T^\sim \varphi = \mathrm{ev}_0$, then $0 \le \varphi(Tx) = x(0) < 0$ ↯

Let X be a pre-Riesz space. For all positive linear functionals $f : X \to \mathbb{R}$, we have:

$$\begin{array}{ccc} f \text{ Riesz homomorphism} & \Longleftarrow & f^{\sim} \text{ interval preserving} \\ \not\Uparrow \; \Downarrow & & \\ f \text{ Riesz* homomorphism} & \xRightarrow{\;i_X^{\sim} \text{ interval preserving}\;} & f^{\sim} \text{ interval preserving} \end{array}$$

$\Longrightarrow$ If $i_X^{\sim}$ is interval preserving, then every Riesz* functional on X is a Riesz homomorphism.

First counterexample: A different point of view

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In the Namioka space X, the linear functional ev_1 is a Riesz* homomorphism that is not a Riesz homomorphism.

$\implies$ $i_X^{\sim} : X \to X^{\rho}$ is a (complete) Riesz homomorphism, but $i_X^{\sim}$ is not interval preserving.

Second counterexample: Four-ray cone

Consider the *four-ray cone*

$$K = \operatorname{pos}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Then

$$i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} : (X, K) \to (\mathbb{R}^4, \mathbb{R}^4_+)$$

is the embedding $(\mathbb{R}^3, K)$ into its Riesz completion, thus a (complete) Riesz homomorphism, but $i^\sim$ is not interval preserving.

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Idea: Take $z := (1, 0, 0, 1)^\top$. For all $y \in [0, z]$, we have $y_2 = y_3 = 0$, thus $i^\sim(y) = (y_1 - y_4, y_1 - y_4, y_1 + y_4)^\top$. Then $w := (1, -1, 1)^\top \in [0, i^\sim(z)]$ but there is no $y \in [0, z]$ with $i^\sim(y) = w$.

Second counterexample: A different point of view

Theorem (B., Kalauch, Stennder, van Gaans, 2025)

Let (X, K) *be a pre-Riesz space with Riesz completion* (X^ρ, i_X). *If* $i_X^\sim$ *is interval preserving, then* $X^\sim$ *has the Riesz decomposition property.*

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Suppose that $\mathbf{dim}(X) < \infty$ and X is Archimedean. Then:

$$X^\sim \text{has the Riesz decomposition property} \iff X \text{ is a Riesz space.}$$

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Therefore:

$$i_X^\sim \text{ is interval preserving} \iff X \text{ is a Riesz space.}$$

Norm-dense subspaces of $\mathrm{C}(\Omega)$

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- X is also order dense in $\mathrm{C}(\Omega)$, thus a pre-Riesz space with a Riesz completion (X^ρ, i_X) satisfying $X^\rho \subseteq \mathrm{C}(\Omega)$ and $i_X(x) = x$ for all $x \in X$.

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- X is also norm dense in X^ρ, thus $i_X' \colon (X^\rho)' \to X'$ is an isomorphism of normed spaces

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- X is also norm dense in X^ρ, thus $i'_X \colon (X^\rho)' \to X'$ is an isomorphism of normed spaces
- With Kantorovich's extension theorem, one can show that i'_X is also bipositive. Thus i'_X is an order isomorphism.

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- Note: $X' = X^\sim$ and $(X^\rho)' = (X^\rho)^\sim$

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- Note: $X' = X^\sim$ and $(X^\rho)' = (X^\rho)^\sim$

$\Longrightarrow$ $i_X^\sim = i_X'$ is an order isomorphism, thus interval preserving

A final question

Let (X, K) be a pre-Riesz space with Riesz completion (X^ρ, i_X). If $i_X^\sim$ is interval preserving, then:

- Every Riesz* functional on X is a Riesz homomorphism.
- $X^\sim$ has the Riesz decomposition property.

A final question

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Is the converse also true?

Thank you!

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