Pervasiveness of $\mathcal{L}^r(E, F)$ in $\mathcal{L}^r(E, F^\delta)$

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Positivity XI

June 9, 2025

This talk is based on a joint article with Quinn Kiervin Starkey that has been accepted for publication in *Indagationes Mathematicae*.

What is this talk about?

The space of all regular operators $\mathcal{L}^r(E,F)$ between Archimedean Riesz spaces E and F has been extensively studied when the range space F is order complete. A wealth of results is available concerning the order structure of $\mathcal{L}^r(E,F)$ in this context.

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Let E, F be Riesz spaces[1]*.* **If F is order complete**, *then $\mathcal{L}^{oc}(E, F)$ is a band of $\mathcal{L}^r(E, F)$.*

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Theorem (Onno Van Gaans & Anke Kalauch, 2008)

$\mathcal{L}^{oc}(\ell_0^\infty, \ell_0^\infty)$ *is a band of* $\mathcal{L}^r(\ell_0^\infty, \ell_0^\infty)$.

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Theorem (Anthony Wickstead, 2024)

If F is almost Dedekind σ-complete, then $\mathcal{L}^{oc}(\ell_0^\infty,F)$ is a band of $\mathcal{L}^r(\ell_0^\infty,F)$.

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Preliminaries

Partially ordered vector spaces

KvG Pre-Riesz Spaces by Anke Kalauch and Onno van Gaans.

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Fix a partially ordered vector space $(X, \geq)$, a subspace Y of X, a subset A of X,

- X is *directed* if for every $x \in X$, there exists $\widetilde{x} \in X$ such that $\widetilde{x} \geq x, 0$.
- A is *pervasive* in X if for every $x \in X$ with $x > 0$ there exists $y \in Y$ such that $0 < y \leq x$.
- Y is said to be *majorizing* in X if for every $x \in X$ there is $y \in Y$ such that $y \geq x$.
- Y is *order dense* in X if $x = \sup\{y \in Y \mid y \leq x\} = \inf\{y \in Y \mid y \geq x\}$ for all $x \in X$.
- $A^u = \{x \in X \mid x \geq a \text{ for all } a \in A\}$.
- We write $x \perp y$ if $\{x + y, -x - y\}^u = \{x - y, -x + y\}^u$ $\forall x, y \in X$.
- The *disjoint complement* Y^d of Y in X is the set $Y^d = \{x \in X \mid x \perp y \text{ for all } y \in Y\}$.
- Y is a *band* of X if $Y^{dd} = Y$.

Preliminaries

Regular operators

Let E, F be Riesz spaces.

- $\mathcal{L}^r(E,F)$ is the space of all regular operators from E to F (i.e. operators that can be written as a difference of positive linear operators).
- $\mathcal{L}^b(E,F)$ is the space of all order bounded operators from E to F.
- A net $(x_\alpha)_{\alpha\in A}$ in X is said to *converge in order* to $x \in X$, written $x_\alpha \xrightarrow{o} x$ if there exists another net $(b_\gamma)_{\gamma\in\Lambda}$ in X such that $b_\gamma \downarrow 0$ and for any $\gamma \in \Lambda$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq b_\gamma$ for all $\alpha \geq \alpha_0$.
- For a net $(x_\alpha)_{\alpha\in A}$ in X and $x \in X$ we write $x_\alpha \xrightarrow{o_1} x$ if there exists a net $(y_\alpha)_{\alpha\in A}$ in X such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$
- We say that a linear operator $T : E \to F$ is order continuous if for any net $(x_\alpha)_{\alpha\in A}$ in E such that $x_\alpha \xrightarrow{o} 0$ we have $T(x_\alpha) \xrightarrow{o} 0$. We denote the space of all order-continuous operators between E and F by $\mathcal{L}^{oc}(E,F)$.

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Throughout this talk, we will view F as a sublattice of its Dedekind completion F^δ and thus $\mathcal{L}^r(E,F)$ is a subspace of $\mathcal{L}^r(E,F^\delta)$

Pervasiveness as a condition for answering Q1 affirmatively

Theorem

If $\mathcal{L}^r(E,F)$ is pervasive in $\mathcal{L}^r(E,F^\delta)$, then $\mathcal{L}^{oc}(E,F) \cap \mathcal{L}^r(E,F)$ is a band of $\mathcal{L}^r(E,F)$.

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Sketch Proof.

Let $\mathcal{S}$ be the Riesz subspace of $\mathcal{L}^r(E,F^\delta)$ generated by $\mathcal{L}^r(E,F)$.

- We can show $\mathcal{L}^r(E,F)$ is majorizing in $\mathcal{S}$. Therefore $\mathcal{L}^r(E,F)$ is order dense in $\mathcal{S}$ (see [KvG]).

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- By applying Ogasawara's theorem we have that $\mathcal{L}^{oc}(E,F^\delta)$ is a band of $\mathcal{L}^r(E,F^\delta)$. We can then show that $\mathcal{B}=\mathcal{S}\cap\mathcal{L}^{oc}(E,F^\delta)$ is a band of $\mathcal{S}$.

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- By [KvG] we get $\mathcal{B}\cap\mathcal{L}^r(E,F)=\mathcal{L}^{oc}(E,F^\delta)\cap\mathcal{L}^r(E,F)$ is a band of $\mathcal{L}^r(E,F)$.

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- By [KvG] we get $\mathcal{B}\cap\mathcal{L}^r(E,F)=\mathcal{L}^{oc}(E,F^\delta)\cap\mathcal{L}^r(E,F)$ is a band of $\mathcal{L}^r(E,F)$.
- Finally, since for any net (x_α) in F we have $x_\alpha \xrightarrow{o} 0$ in F iff $x_\alpha \xrightarrow{o} 0$ in F^δ (see [Abramovich & Sirotkin 2005]) we get $\mathcal{L}^{oc}(E,F^\delta)\cap\mathcal{L}^r(E,F)=\mathcal{L}^{oc}(E,F)\cap\mathcal{L}^r(E,F)$.

□

When $\mathcal{L}^r(E,F)$ is pervasive in $\mathcal{L}^r(E,F^\delta)$?

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$$\mathcal{RO}(E,F) = \{f \otimes v \mid f \in E^\sim, v \in F\}$$

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Let $T \in \mathcal{L}^r(E,F^\delta)$ with $T > 0$. Then we may find $f \in E^\sim, v \in F^\delta$ such that $0 < f \otimes v \leq T$.

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Let $T \in \mathcal{L}^r(E,F^\delta)$ with $T > 0$. Then we may find $f \in E^\sim, v \in F^\delta$ such that $0 < f \otimes v \leq T$. In fact we can assume without loss of generality that $f > 0$ and $v \in F^\delta_+ \setminus \{0\}$. Now by the pervasiveness of F in F^δ we may choose $v_0 \in F$ such that $0 < v_0 \leq v$ and put $S = f \otimes v_0$, then clearly $S \in \mathcal{L}^r(E,F)$ and $0 < S \leq f \otimes v \leq T$. $\square$

A sufficient condition on the range space F

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Let $\{e_i \mid i \in I\}$ be a complete disjoint system of atoms in F^δ and $T \in \mathcal{L}^r(E, F^\delta)$, $T > 0$. Then we have

$$T(x) = \bigvee_{i \in I} \lambda_{e_i}(T(x))e_i, x \in E_+.$$

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Pick $x_0 \in E_+$ and $j \in I$ such that $\lambda_{e_j}(T(x_0)) > 0$. Put $\widetilde{T} = (\lambda_{e_j} \circ T) \otimes e_j$, then we have $\widetilde{T} \in \mathcal{RO}(E, F^\delta)$ and $0 < \widetilde{T} \leq T$. Therefore $\mathcal{RO}(E, F^\delta)$ is pervasive in $\mathcal{L}^r(E, F^\delta)$. $\square$

A sufficient condition on the domain space E

Lemma

Let F be an order complete Riesz space. Let $\{e_i \mid i \in I\}$ be a complete disjoint system of atoms in E, and let λ_{e_i} denote the coordinate functional of e_i. Let $T \in \mathcal{L}^r(E,F)_+$, P be the band projection associated with $\mathcal{L}^{oc}(E,F)$. Then $\sum_{i\in\alpha} \lambda_{e_i}(x) \otimes T(e_i) \uparrow_{\alpha\in Fin(I)} P(T)(x)$ for all $x \in E_+$.

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Proposition

If E is atomic and the uniform closure of the span of its atoms is equal to E or has codimension 1, then $\mathcal{L}^r(E,F)$ is pervasive in $\mathcal{L}^r(E,F^\delta)$.

Sketch proof.

If $T(e_j) > 0$ for some $j \in I$ then put $\widetilde{T} = \lambda_{e_j} \otimes T(e_j) \in \mathcal{RO}(E,F^\delta)$. By the above Lemma we have $0 < \widetilde{T} \leq P(T) \leq T$.

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Example

If $E = \ell_0^\infty$ or $E = c$, then $\mathcal{L}^r(E,F)$ is pervasive in $\mathcal{L}^r(E,F^\delta)$.

Back to the case $\mathcal{L}^r(\ell_0^\infty, F)$

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Theorem

$\mathcal{L}^{oc}(\ell_0^\infty, F)$ is a band of $\mathcal{L}^r(\ell_0^\infty, F)$. (No assumptions on F)

Proof.

By (Abramovich & Wickstead, 1991) we have $\mathcal{L}^b(\ell_0^\infty, F) = \mathcal{L}^r(\ell_0^\infty, F)$ and by (Abramovich & Sirotkin, 2005) we have that every order-continuous operator is order-bounded. Thus

$$\mathcal{L}^{oc}(\ell_0^\infty, F) \cap \mathcal{L}^r(\ell_0^\infty, F) = \mathcal{L}^{oc}(\ell_0^\infty, F) \cap \mathcal{L}^b(\ell_0^\infty, F) = \mathcal{L}^{oc}(\ell_0^\infty, F).$$

By the previous results it now follows that $\mathcal{L}^{oc}(\ell_0^\infty, F)$ is a band of $\mathcal{L}^r(\ell_0^\infty, F)$. □

When $\mathcal{L}^{oc}(\ell_0^\infty, F)$ is directed?

Theorem (Wickstead 2024)

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Example ($\mathcal{L}^{oc}(\ell_0^\infty, F)$ may not be directed)

Let Γ be an uncountable set equipped with the discrete topology, and let $K = \Gamma \cup \{\infty\}$ be the one-point compactification of Γ. We put $F = C(K)$, the Banach lattice of continuous real-valued functions on K. Then $\mathcal{L}^{oc}(\ell_0^\infty, C(K))$ is not directed. The example is based on a construction (attributed to Fremlin) of a sequence (x_n) in $C(K)$ such that $x_n \xrightarrow{o} 0$ but $x_n \not\xrightarrow{o_1} 0$.

$\mathcal{L}^{oc}(E,F)$ *may not be a subset of* $\mathcal{L}^{r}(E,F)$.

Example

Let E_K denote the space of all double sequence $(x_{n,m})_{n,m\in\mathbb{N}}$ such that

(i) There exists $n_0 \in \mathbb{N}$ such that $x_{n,m} = x_{\widetilde{n},\widetilde{m}}$ for all $n, \widetilde{n} \geq n_0$ and $m, \widetilde{m} \in \mathbb{N}$.

(ii) For all $n \in \mathbb{N}$ we have that $(x_{n,m})_{m\in\mathbb{N}} \in \ell_0^\infty$.

We denote by $\ell_0^\infty(\mathbb{N}\times\mathbb{N})$ the space of double sequences which are constant except on a finite set. Let $T : E_K \to \ell_0^\infty(\mathbb{N}\times\mathbb{N})$ given by the formula

$$(Tx)_{n,m} = x_{n,2m-1} - x_{n,2m}$$

In [Abramovich & Wickstead 1991] it is proved that T is an order bounded operator that is not regular. We can also show that T is order continuous.

The Riesz-Kantorovich property

We say that $\mathcal{L}^r(E,F)$ has the **Riesz-Kantorovich property** whenever for each $T \in \mathcal{L}^r(E,F)$ such that T^+ exists in $\mathcal{L}^r(E,F)$ we have $T^+(x) = \sup\{T(x) \mid y \in [0,x]\}$ for each $x \in E_+$.

Q2: What happens if we drop the assumption that F is order complete?Does $\mathcal{L}^{oc}(E,F)$ has the Riesz-Kantorovich property?

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Theorem (Michael Elliott 2019)

There exists a compact Hausdorff space K such that $\mathcal{L}^r(L^1[0,1], C(K))$ does not have the Riesz-Kantorovich property.

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Theorem (Abramovich & Wickstead 1991)

If E is uniformly complete, then $\mathcal{L}^r(E, \ell_0^\infty)$ has the Riesz-Kantorovich property.

Pervasiveness (again!) as a condition for answering Q2 affirmatively

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If $\mathcal{L}^r(E, F)$ is pervasive in $\mathcal{L}^r(E, F^\delta)$, then $\mathcal{L}^r(E, F)$ has the Riesz-Kantorovich property.

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Sketch proof.

Let $T \in \mathcal{L}^r(E, F)$ such that T^+. Take $\mathcal{S}$ be the sublattice generated by $\mathcal{L}^r(E, F)$ in $\mathcal{L}^r(E, F^\delta)$, then $\mathcal{L}^r(E, F)$ is order dense in $\mathcal{S}$. By [KvG] it follows that T^+ can be calculated in $\mathcal{S}$, thus T^+ is given by the Riesz Kantorovich formula.

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If $\mathcal{L}^r(E, F)$ is pervasive in $\mathcal{L}^r(E, F^\delta)$, then $\mathcal{L}^r(E, F)$ has the Riesz-Kantorovich property.

Sketch proof.

Let $T \in \mathcal{L}^r(E, F)$ such that T^+. Take $\mathcal{S}$ be the sublattice generated by $\mathcal{L}^r(E, F)$ in $\mathcal{L}^r(E, F^\delta)$, then $\mathcal{L}^r(E, F)$ is order dense in $\mathcal{S}$. By [KvG] it follows that T^+ can be calculated in $\mathcal{S}$, thus T^+ is given by the Riesz Kantorovich formula.

□

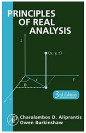
Corollary

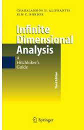
$\mathcal{L}^r(E, \ell_0^\infty)$ has the Riesz-Kantorovich property. (No assumptions on E).

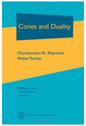
Open questions

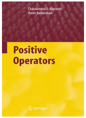
- Is $\mathcal{L}^{oc}(E,F) \cap L^r(E,F)$ always a band of $L^r(E,F)$?
- Is $\mathcal{L}^r(E,F)$ pervasive in $L^r(E,F^\delta)$ if E is atomic?

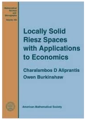
In Memory of C. D. Aliprantis

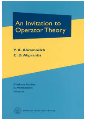












Let me close with a personal note. Like many of you, I have been deeply influenced by the work of Professor C.D. Aliprantis. His contributions to mathematics and economic theory have shaped the way we think, teach, and write. But beyond the theorems and textbooks, what stays with me is the profound dedication he brought to his work—a dedication that spanned decades and left a lasting impact on our field.

Years ago with the encouragement of my advisor Ioannis Polyrakis, I wrote to Professor C. D. Aliprantis. He replied:

"Thank you very much for your letter. It is always nice to read a letter like yours. I have spent my whole scientific career on scholar activities. Each of the books you mention took a minimum of ten years to write it. Unfortunately, my health has failed me and I can no longer work... I am very sad about this."

We remember him with deep gratitude and respect.