On the Transfer of Completeness and Projection Properties in Truncated Vector Lattices

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POSITIVITY XII

Plan

Introduction :

Truncated Riesz spaces were introduced by D.H. Fremlin in 1974 as Riesz subspaces of $\mathbb{R}^X$ satisfying *Stone's condition*; that is, for every non-negative function f in the space, the truncated function $1 \wedge f$ (where 1 denotes the constant function equal to 1) also belongs to the space.

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Definition

A *truncation*, that is, a nonzero map $x \to x^*$ from the positive cone E^+ into itself such that :

$(\tau_1)x^* \wedge y = x \wedge y^*$ for all $(x, y) \in E^+ \times E^+$.
(τ_2)If $0 \leq x \in E$ *and* $x^* = 0$ then $x = 0$.

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- *The direct sum $E \oplus \mathbb{R}$ equipped with the positive cone*

$$[E \oplus \mathbb{R}]^+ = E^+ \cup \left\{ x + \alpha : \alpha \in]0, +\infty[\ et \ \left(\frac{1}{\alpha}x^-\right)^* = \frac{1}{\alpha}x^- \right\}$$

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Archimedean Property

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A truncation satisfying the additional axiom

$$(\tau_3) \text{ If } 0 \leq x \in E \text{ and } (nx)^* = nx \ \forall n \in \{1, 2, ...\} \quad \text{then} \quad x = 0.$$

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Example (1)

If we consider the plan $E = \mathbb{R}^2$, equipped with its lexicographic ordering, it's well known that E is not Archimedean Riesz space, yet the truncation defined on E by $\overline{(x, y)} = (x, y) \wedge (0, 1)$, for all $(x, y) \in E^+$ is an Archimedean truncation on E.

Archimedean Property

Example (2)

But also, on the other hand, if $E = \{0\} \times \mathbb{R}$, then *obviously*, *E is an* Archimedean vector sublattice of $\mathbb{R}^2$.
Define a truncatíon on E by setting :

$$\overline{(0, r)} = (0,\ r) \wedge (1, 0) = (0,\ r) \text{ for all } r \in \mathbb{R}$$

(*the infimum is taken in* $\mathbb{R}^2$). *Observe that if* $n \in \{1, 2, ...\}$,*then*

$$\overline{n(0, r)} = n(0,\ r) \text{ for all } r \in \mathbb{R}.$$

Hence this truncation is not an Archimedean truncation on E

Archimedean Property

Theorem (HABIBI M. , HAFSI H. 2025)

Let E be a truncated Riesz space. Then $E \oplus \mathbb{R}$ is Archimedean if and only if E is Archimedean and the truncation is an Archimedean truncation.

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The result above leads to a simpler characterization in the unital case.

Corollary (HABIBI M. , HAFSI H. 2025)

Let E be a unital truncated Riesz space. Then $E \oplus \mathbb{R}$ is Archimedean if and only if E is Archimedean.

The following theorem relates the set of fixed points of the truncation to its unit element.

Theorem (HABIBI M. , HAFSI H. 2025)

Let E be a non unital truncated Riesz space such that $E \oplus \mathbb{R}$ is Archimedean.

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Theorem (HABIBI M. , HAFSI H. 2025)

Let E be a non unital truncated Riesz space such that $E \oplus \mathbb{R}$ is Archimedean.For every $x \in E \oplus \mathbb{R}$ with $x > 0$,
$\sup_{E\oplus\mathbb{R}} \{\bar{y}, y \in E, 0 \leq y \leq x\} = \bar{x}$.

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In particular, $\sup \{\bar{x}, x \in E^{+}\} = 1$.

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Relatively Uniformly Complete

Definitions

1. Given $u \in E^+$, it is said that the sequence (x_n) in E converges u-uniformly to the element $x \in E$, or that x is an u-uniform limit for (x_n), whenever, for every $\epsilon > 0$, there exists a natural number N_ϵ such that $| x_n - x | \leq \epsilon u$ holds for all $n \geq N_\epsilon$.

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3. The sequence (x_n) in E is an u-uniform Cauchy sequence with $u > 0$ if, for every $\epsilon > 0$, we have $\mid x_n - x_m \mid \leq \epsilon u$ for all n and m sufficiently large.
4. The Riesz space E is called uniformly complete whenever, for every $u \in E^+$, every u-uniform Cauchy sequence has an u-uniform limit.

Relatively Uniformly Complete Property

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1. *If $E \oplus \mathbb{R}$ is relatively uniformly complete, then E is relatively uniformly complete.*
2. *If E is unital and uniformly complete, then $E \oplus \mathbb{R}$ is uniformly complete.*

Example (3)

Let $E = c_{00}$ equipped with the pointwise order. For $u = (u_n)_{n\in\mathbb{N}} \in c_{00}$, let $\bar{u} = u \wedge 1$, where 1 is the constant sequence equal 1.

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We claim that :

1. E is relatively uniformly complete.
2. $E \oplus \mathbb{R}$ is not relatively uniformly complete.

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Dedekind Completeness

Definition

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Lemma (Locally solide)

A Riesz space E is Dedekind complete if and only if for every net (u_α) in E satisfying $0 \leq u_\alpha \uparrow \leq v$ in E we have $u_\alpha \uparrow u$ for some u.

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Lemma (HABIBI M. , HAFSI H. 2025)

Let (x_α) *a net in a Riesz space satisfying* $0 \leq x_\alpha \uparrow \leq u + v$ *, then there exists a net* (u_α) *and* (v_α) *satisfying* $x_\alpha = u_\alpha + v_\alpha$ *,* $0 \leq u_\alpha \uparrow \leq \mid u \mid$ *and* $0 \leq v_\alpha \uparrow \leq \mid v \mid$*.*

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In general, when the suprema exists, the supremum of the sum of two nets may differ from the sum of their individual suprema. However, in the case of increasing nets, the equality holds. This is stated in the next lemma.

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Lemma (HABIBI M. , HAFSI H. 2025)

Let $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$ be two increasing nets indexed by the same directed set A, and suppose that $\sup x_\alpha$ and $\sup y_\alpha$ exist. Then, the supremum of the net $(x_\alpha + y_\alpha)_{\alpha\in A}$ exists, and we have :
$\sup_{\alpha\in A}(x_\alpha + y_\alpha) = \sup_{\alpha\in A} x_\alpha + \sup_{\alpha\in A} y_\alpha$.

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Lemma (HABIBI M. , HAFSI H. 2025)

Let E be a truncated vector lattice and A a non-empty subset of E. If $a_0 = sup(A)$ exists in E, then a_0 is also the supremum of A when considered as a subset of $E \oplus \mathbb{R}$.

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Let E be a truncated vector lattice. Then, $E \oplus \mathbb{R}$ is Dedekind complete if and only if E is Dedekind complete and satisfies the following property

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Corollary (HABIBI M. , HAFSI H. 2025)

Let E be a unital truncated vector lattice. Then E is Dedekind complete if and only if $E \oplus \mathbb{R}$ is Dedekind complete.

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For an order dense Riesz subspace E of an Archimedean Riesz space M we have :
If E is laterally complete, then E majorizes M.

Now, we move on to the statement of the main theorem of this section

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Theorem (HABIBI M. , HAFSI H. 2025)

Let E be a truncated Riesz space.
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(2) *Suppose $E \oplus \mathbb{R}$ is laterally complete and Archimedean. Then, E is laterally complete if and only if E is unital*

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Let E be a truncated Riesz space.

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2. *Assume that $E \oplus \mathbb{R}$ is universally complete. Then E is universally complete if and only if E is unital.*

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3. *There exists an ideal A of E such that $E = B \oplus A$ holds.*

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THANKS

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