Arens Regularity of Banach Lattice Algebras

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- A'' is a Banach algebra with respect the first $(.)$ or the second $(*)$ Arens multiplication.
- For all $x, y \in A$, $x''.y'' = x''*y'' = (xy)''$.

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$$\begin{aligned} F.G(f) &= \lim_\alpha \lim_\beta f(a_\alpha b_\beta) \\ F*G(f) &= \lim_\beta \lim_\alpha f(a_\alpha b_\beta) \end{aligned}$$

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$f \in A'$ is said to be **weakly almost periodic** (**wap**) if the orbit O_f of f where , $O_f = \{f.x : x \in B(A)\}$ is a weakly relatively compact subset in A'.

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$$\mathrm{wap}(A) = \{f \in A',\ f \text{ is a wap}\}$$

Arens regularity

Theorem

(Grothendieck, Pym) For a Banach algebra A, the following are equivalents.

1. *A is Arens regular.*
2. $\mathrm{wap}(A) = A'$.
3. *For all $f \in A'$ and for every sequences (x_n) and $(y_n) \subset B(A)$, we have*

$$\lim_m \lim_n f(x_n y_m) = \lim_n \lim_m f(x_n y_m)$$

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Corollary

Subalgebra and quotient algebra of an Arens regular algebra are also Arens regular.

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3. Let X be a Banach space. The algebra $\mathcal{K}(X)$ is Arens regular if and only if X reflexive. (Young 1976)
4. Any C^*-Banach algebra is Arens regular.

Banach lattices

Definition

A **real Banach lattice** A is a Banach space +vector lattice +

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Classical examples of Banach lattices with order continuous norms include the Lebesgue spaces $\mathcal{L}_p(\mu)$ $(1 \leq p < \infty)$ spaces.

Banach lattice with order continuous norm

Theorem

For a Banach lattice E the following statements are equivalent.

- *E has order continuous norm.*
- *E is σ-Dedekind complete and satisfies: for every sequence (x_n) with $x_n \downarrow 0$ we have $\|x_n\| \longrightarrow 0$.*
- *Each order interval of E is weakly compact.*
- *E is an ideal of E''.*

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- E *is an ideal of* E''.

Theorem

E' *has an order continuous norm* $\Longleftrightarrow$ *Every norm-bounded disjoint sequence in* E *is weakly null.*

Banach lattice algebras

Definition

Let A be a Banach lattice, a **Banach lattice algebra product** on A is a bilinear mapping $p : (x, y) \longrightarrow xy$ such that (A, p) is a Banach algebra and

$$xy \geq 0 \text{ for all } 0 \leq x, y \in A$$

$$\|xy\| \leq \|x\| \|y\| .$$

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Fact

The second dual A'' of a Banach lattice algebra, with either Arens products, is also a Banach lattice algebra.

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Goal: Lattice conditions ensuring Arens regularity of Banach lattice algebras.

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$$L : x \longrightarrow L_x \text{ where } L_x(y) = xy \text{ for all } y$$

is a lattice and algebra homomorphism from A into

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$\Longrightarrow f \in \mathrm{wap}(A)$.

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2. $\mathcal{M}\text{-}wap(A) = A' \Longrightarrow A$ is Arens regular.
3. For AM-space, $\mathcal{M}\text{-}wap(A) = \text{wap}(A)$ for every compatible product on A.

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Examples

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Define the product on ℓ^1 by

$$x.y = (\sum_{n\in\mathbb{N}} x_n)(\sum_{n\in\mathbb{N}} y_n)u \quad \text{where} \quad u = (\frac{1}{2}, \frac{1}{2}, 0, \ldots).$$

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Then $f = (1, -1, 0, ..) \in \mathcal{M}$-wap$(A)$. But $|f| = (1, 1, 0, ..) \notin \mathcal{M}$-wap$(A)$ since $|f| .e_n = e_n$, where (e_n) the standard unit vectors.

M-weakly almost periodic

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In particular, if A' has an order continuous norm, then A is Arens regular with respect any compatible product.

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(2)$\Longrightarrow$(1). Use the fact that If A' has an order continuous norm then A' is KB-space.

Corollary

If A admits a product making it a **Banach** f**-algebra with unit**. *Then, A is Arens regular, for* **every compatible product** *on A.*

Factorization of linear functionals.

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Theorem

Let A be a Banach lattice such that A' has an order continuous norm. Let p be any Banach lattice algebra product on A. Then for each $0 \leq f \in A'$ there exist:

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3. *An M-weakly compact operator $S : A \longrightarrow \Psi_f$ such that for all $x \in A$,*

$$f.x = S(Q(x)).$$

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Arens regularity of the positive tensor product of Banach lattice algebras.

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Definition (Fremlin)

The positive projective tensor product $A \otimes_{|\pi|} B$ is the completion of the algebraic tensor product $A \otimes B$ with respect to the norm

$$\|u\|_{|\pi|} = \sup\{\|\widehat{\varphi}(u)\| : \varphi \in \mathcal{L}\} \text{ for all } u \in A \otimes B,$$

where L is the set of all positive bilinear maps from $A \times B$ to all Banach lattices G with norm ≤ 1 and $\widehat{\varphi} : A \otimes B \longrightarrow G$ is the linear map corresponding to φ.

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(Jaber 2020). Let A and B be Banach lattice algebras. On the positive projective tensor product $A \otimes_{|\pi|} B$ of A and B there exists a natural algebra structure, under which it becomes a Banach lattice algebra

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If $A \otimes_{|\pi|} B$ is Arens regular, then so are A and B.

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Theorem

Let A and B be non trivial Banach lattice algebras.

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2. *If $\mathrm{wap}(A \otimes_{|\pi|} B) = (A \otimes_{|\pi|} B)'$ then $\mathrm{wap}(A) = A'$ and $\mathrm{wap}(B) = B'$*

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Idea: For $f \in A'$ and $g \in B' \Longrightarrow f \otimes g \in (A \otimes_{|\pi|} B)'$ where

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$$\begin{aligned} 0 &= \lim_n f \otimes g((x_n \otimes \frac{b_1}{\|b_1\|})(y_n \otimes \frac{b_2}{\|b_2\|}) \\ &= \lim_n f(x_n y_n) g(b_1 b_2). \end{aligned}$$

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The positive tensor product of two Arens regular Banach lattice algebras need not be regular.

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Example

Let $A = c_0 \oplus \mathbb{R}$ equipped with the multiplication defined by

$$(f, \alpha).(g, \beta) = (fg + \alpha f + \beta g, \alpha\beta)$$

the norm and the order on A are given by

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$A \otimes_{|\pi|} c_0$ is not Arens regular.

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Then,

$$\varphi((f_n, 1) \otimes f_n).(0, 1) \otimes g_m) = \sum_{k=1}^{n} \frac{1}{k} g_m(k) = \begin{cases} \frac{(m+1)}{2m} & \text{if } n \geq m \\ \frac{n(n+1)}{2m^2} & \text{if } n < m \end{cases}$$

Thus,

$$\lim_n \lim_m \varphi((f_n, 1) \otimes f_n).(0, 1) \otimes g_m) = 0.$$

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Corollary

Let A and B two Banach lattices such that $\mathcal{L}^r(A, B')$ has an order continuous norm then A, B and $A \otimes_{|\pi|} B$ are Arens regular .

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Corollary

Let A and B two Banach lattices such that $\mathcal{L}^r(A, B')$ has an order continuous norm then A, B and $A \otimes_{|\pi|} B$ are Arens regular .

Example

Let A be Banach AM-algebra with unit element e and B a Banach lattice such that B' has an order continuous norm. Then $A \otimes_{|\pi|} B$ is Arens regular.

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(Jaber 2020) The positive projective tensor product of two Banach f-algebras is a Banach f-algebra

Example

$A \otimes_{|\pi|} B$ is Arens regular whenever A and B are Banach f-algebras.

Final remark

Let A and B be two Banach lattice algebras.

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- There exist a unique continuous algebra homomorphism $\theta : A \otimes_\pi B \longrightarrow A \otimes_{|\pi|} B$ that extend the identity on the algebraic tensor product $A \otimes B$.

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- answer is negative, in general.

Example

Consider compact Hausdorff spaces K and K', each containing a perfect set (i.e., closed set without isolated points). Take $A = C(K)$ and $B = C(K')$. Then the positive projective tensor product $A \otimes_{|\pi|} B$ is Arens regular as it is an f-algebra.
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