On the order bidual of $C(X)$

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The bidual of $\mathrm{C}(K)$

An old result

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Let K be a compact Hausdorff space. There exists a unique compact Hausdorff space $\widetilde{K}$ so that $\mathrm{C}(K)^{**}$ is isometrically lattice (and ring) isomorphic to $\mathrm{C}(\widetilde{K})$.

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Definition

Let K be a compact Hausdorff space. We call $\widetilde{K}$ the hyper-Stonean cover of K.

Question

Can we replace compact K with realcompact X, and the norm (bi)dual with the order (bi)dual?

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Realcompact space: A Tychonoff space which is a closed subspace of some power of $\mathbb{R}$.

For every topological space X there is a unique realcompact κX so that $\mathrm{C}(X)$ & $\mathrm{C}(\kappa X)$ are (ring and lattice) isomorphic.

Preliminaries

Order Adjoints

Theorem

$T : \mathrm{E} \to \mathrm{F}$ a positive operator; $T^{\sim} : \mathrm{F}^{\sim} \to \mathrm{E}^{\sim}$ its order adjoint, $\varphi \mapsto \varphi \circ T$.

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(iii) *T interval preserving $\Rightarrow T^{\sim}$ a lattice homomorphism.*

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(v) *$T^{\sim}$ lattice homomorphism $\Rightarrow$ T interval preserving if ${}^{\circ}\mathrm{F}^{\sim} = \{0\}$.*

Categories of Vector Lattices

	Objects	Morphisms
VL	Vector lattices	Lattice homomorphisms
NVL	Vector lattices	Normal lattice homomorphisms
IVL	Vector lattices	Interval preserving lattice homomorphisms
NIVL	Vector lattices	Normal, interval preserving lattice homomorphisms
TOP	Topological spaces	Continuous functions

Direct Limits

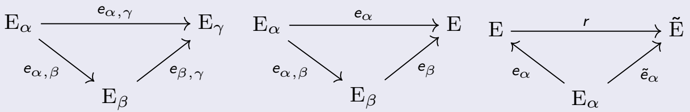
Definitions

Definition

Let **C** *be a category, I a directed set,* E_α *a vector lattice for each* $\alpha \in I$*, and* $e_{\alpha,\beta} : \mathrm{E}_\alpha \to \mathrm{E}_\beta$ *a* **C***-morphism for all* $\alpha \preccurlyeq \beta$ *in* I*.*

- $\mathcal{D} := \big((\mathrm{E}_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta}\big)$ *is a direct system in* **C** *if, for all* $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ *in* I*,*
$$e_{\beta,\gamma} \circ e_{\alpha,\beta} = e_{\alpha,\gamma}.$$
- $\mathcal{S} := (\mathrm{E}, (e_\alpha)_{\alpha \in I})$ *is a compatible system of* $\mathcal{D}$ *in* **C** *if, for all* $\alpha \preccurlyeq \beta$ *in* I*,*
$$e_\beta \circ e_{\alpha,\beta} = e_\alpha.$$
- *The compatible system* $\mathcal{S} := (\mathrm{E}, (e_\alpha)_{\alpha \in I})$ *of* $\mathcal{D}$ *in* **C** *is the direct limit of* $\mathcal{D}$ *if for any compatible system* $\tilde{\mathcal{S}} := (\tilde{\mathrm{E}}, (\tilde{e}_\alpha)_{\alpha \in I})$ *of* $\mathcal{D}$ *in* **C** *there exists a unique* **C***-morphism* $r : \mathrm{E} \to \tilde{\mathrm{E}}$ *so that, for every* $\alpha \in I$*,*
$$r \circ e_\alpha = \tilde{e}_\alpha.$$

$$\mathrm{E} = \varinjlim \mathrm{E}_\alpha$$



Example

Example (de Jeu & vdWalt, 2024)

X a realcompact space
– $\mathfrak{K}_X$ all nonempty compact subsets of X, ordered by inclusion;
– $\mathfrak{C} = \{K_\alpha\}_{\alpha \in I}$ cofinal in $\mathfrak{K}_X$.

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- $\varinjlim \mathrm{M}(K_\alpha) = \mathrm{M}_c(X)$ in **NIVL**.

Equivalently: Identify $\mathrm{M}(K_\alpha)$ with the band $\{\mu \in \mathrm{M}_c(X) \ : \ S_\mu \subseteq K_\alpha\}$ in $\mathrm{M}_c(X)$ & $T_{\alpha,\beta}$, T_α with inclusions. Still, $\varinjlim \mathrm{M}(K_\alpha) = \mathrm{M}_c(X) = \mathrm{C}(X)^\sim$.

Inverse Limits

Definitions

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Let **C** *be a category, I a directed set, E_α a vector lattice for each $\alpha \in I$, and $p_{\beta,\alpha} : \mathrm{E}_\beta \to \mathrm{E}_\alpha$ a* **C***-morphism for all $\beta \succcurlyeq \alpha$ in I.*

- $\mathcal{I} \coloneqq \big((\mathrm{E}_\alpha)_{\alpha \in I}, (p_{\beta,\alpha})_{\beta \succcurlyeq \alpha}\big)$ *is an inverse system in* **C** *if, for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I,*

$$p_{\beta,\alpha} \circ p_{\gamma,\beta} = p_{\gamma,\alpha}.$$

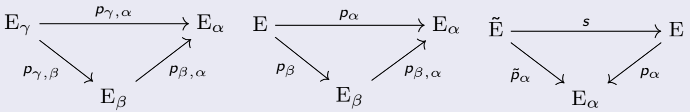
- $\mathcal{S} \coloneqq (\mathrm{E}, (p_\alpha)_{\alpha \in I})$ *is a compatible system of $\mathcal{I}$ in* **C** *if, for all $\alpha \preccurlyeq \beta$ in I,*

$$p_{\beta,\alpha} \circ p_\beta = p_\alpha.$$

- *The compatible system $\mathcal{S} \coloneqq (\mathrm{E}, (p_\alpha)_{\alpha \in I})$ of $\mathcal{I}$ in* **C** *is the inverse limit of $\mathcal{I}$ if for any compatible system $\tilde{\mathcal{S}} \coloneqq (\tilde{\mathrm{E}}, (\tilde{p}_\alpha)_{\alpha \in I})$ of $\mathcal{I}$ in* **C** *there exists a unique* **C***-morphism $s : \tilde{\mathrm{E}} \to \mathrm{E}$ so that, for every $\alpha \in I$,*

$$p_\alpha \circ s = \tilde{p}_\alpha.$$

$$\mathrm{E} = \varprojlim \mathrm{E}_\alpha$$



Example

Example

Let $\mathcal{D} := \big((X_\alpha)_{\alpha\in I}, (\theta_{\alpha,\beta})_{\alpha\leq\beta}\big)$ be a direct system in **TOP** with direct limit $\mathcal{I} := (X, (\theta_\alpha)_{\alpha\in I})$.

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- For all $\beta \geq \alpha$ in I, define

$$T_{\beta,\alpha} : \mathrm{C}(X_\beta) \ni u \mapsto u \circ \theta_{\alpha,\beta} \in \mathrm{C}(X_\alpha)$$

  and

$$T_\alpha : \mathrm{C}(X) \ni u \mapsto u \circ \theta_\alpha \mathrm{C}(X_\alpha).$$

- Define

$$\mathcal{D}^\star := \left((\mathrm{C}(X_\alpha))_{\alpha\in I}, (T_{\beta,\alpha})_{\beta\geq\alpha}\right)$$

  and

$$\mathcal{I}^\star := (\mathrm{C}(X), (T_\alpha)_{\alpha\in I}).$$

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  and

$$\mathcal{I}^\star := (\mathrm{C}(X), (T_\alpha)_{\alpha\in I}).$$

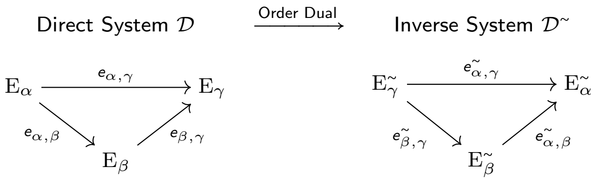
Then

(i) $\mathcal{D}^\star$ is an inverse system in **VL**.

(ii) $\mathcal{I}^\star = \varprojlim \mathcal{D}^\star$ in **VL**.

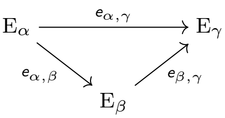
Duality

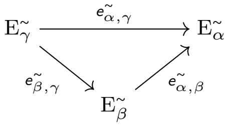
Dual Systems of Direct Systems



Dual Systems of Direct Systems

Direct System $\mathcal{D}$ $\xrightarrow{\text{Order Dual}}$ Inverse System $\mathcal{D}^{\sim}$





Inverse System $\xrightarrow{\text{Order Dual}}$ Direct System

Duality for Direct Limits

Theorem (v Amstel & vdWalt, 2024)

Let $\mathcal{D} \coloneqq \left((\mathrm{E}_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta}\right)$ *be a direct system in* **IVL**, *and* $\varinjlim \mathrm{E}_\alpha = \mathrm{E}$ *in* **IVL**. *Then* $\varprojlim {\mathrm{E}_\alpha}^{\sim} = \mathrm{E}^{\sim}$ *in* **NVL**.

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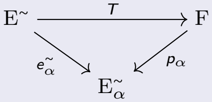

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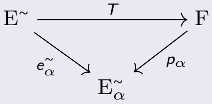

$$\mathrm{E} = \varinjlim \mathrm{E}_\alpha \Rightarrow \mathrm{E}^\sim = \varprojlim \mathrm{E}_\alpha^\sim$$

The space $\mathrm{C}(X)^{\sim\sim}$

The order dual of $C(X)$

Theorem (Hewitt 1950, Gould & Mahowald 1962, v Amstel & vdWalt 2024, de Jeu & vdWalt 2024)

Let X be realcompact. Then

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(i) $\mathrm{C}(X)^{\sim} = \mathrm{M_c}(X) = \varinjlim \mathrm{M}(K_\alpha)$ *in* **NIVL**.

(ii) *Then* $\mathrm{C}(X)^{\sim\sim} = \varprojlim \mathrm{M}(K_\alpha)^{\sim}$ *in* **NVL**.

The set-up

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Let $K_\alpha \subseteq K_\beta \subseteq X$ be compact.

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- $T^{\sim}_{\alpha,\beta} : \mathrm{M}(K_\beta)^{\sim} \to \mathrm{M}(K_\alpha)^{\sim}$ is a normal interval preserving lattice homomorphism.

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- $T^{\sim}_{\alpha,\beta} : \mathrm{C}(\tilde{K}_\beta) \to \mathrm{C}(\tilde{K}_\alpha)$ is a normal interval preserving lattice homomorphism.
- $T^{\sim}_{\alpha,\beta}(\mathbf{1}_{\tilde{K}_\beta}) = \mathbf{1}_{\tilde{K}_\alpha}$.

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- $T^{\sim}_{\alpha,\beta} : \mathrm{C}(\tilde{K}_\beta) \to \mathrm{C}(\tilde{K}_\alpha)$ is a normal interval preserving lattice homomorphism.
- $T^{\sim}_{\alpha,\beta}(\mathbf{1}_{\tilde{K}_\beta}) = \mathbf{1}_{\tilde{K}_\alpha}$.
- There exists $\theta_{\alpha,\beta} : \tilde{K}_\alpha \to \tilde{K}_\beta$ continuous s.t. $T^{\sim}_{\alpha,\beta}(u) = u \circ \theta_{\alpha,\beta}$, $u \in \mathrm{C}(\tilde{K}_\beta)$.

The order bidual of $\mathrm{C}(X)$

Theorem (de Jeu & vdWalt, 2024)

X a realcompact space; $\mathfrak{C} = \{K_\alpha\}_{\alpha\in I}$ cofinal in $\mathfrak{K}_X$.

- $((\tilde{K}_\alpha)_{\alpha\in I}, (\theta_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ *is an direct system in* **TOP**.
- $\varinjlim \tilde{K}_\alpha$ *exists in* **TOP**.
- *If* $Y = \varinjlim \tilde{K}_\alpha$ *in* **TOP** *then* $\varprojlim \mathrm{C}(\tilde{K}_\alpha) = \mathrm{C}(Y)$ *in* **VL**.
- *Let* $\widetilde{X} \coloneqq \kappa Y$. *Then* $\mathrm{C}(\widetilde{X}) = \mathrm{C}(Y) = \varprojlim \mathrm{C}(\tilde{K}_\alpha) = \varprojlim \mathrm{M}(K_\alpha)^\sim = \mathrm{C}(X)^{\sim\sim}$.

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Theorem (De Jeu & vdWalt 2024)

Let X be a realcompact space. There exists a unique realcompact, extremally disconnected space $\widetilde{X}$ so that $\mathrm{C}(X)^{\sim\sim}$ is lattice isomorphic to $\mathrm{C}(\widetilde{X})$.

The space $\widetilde{X}$

Some properties of $\widetilde{X}$

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Let X be a realcompact space. The following statements are true.

(i) *$\widetilde{X}$ is hyper-disconnected.*

(ii) *For every $K \in \mathfrak{K}_X$, there exists a continuous map $\eta_K : \tilde{K} \to \widetilde{X}$ which is a homeomorphism onto its range.*

Some properties of $\widetilde{X}$

Theorem (de Jeu & vdWalt 2024)

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(iv) *There exists a continuous surjection $\pi_X : \widetilde{X} \to X$ so that the canonical embedding $\sigma_X : \mathrm{C}(X) \to \mathrm{C}(\widetilde{X})$ is given by $\sigma_X(u) = u \circ \pi_X$, $u \in \mathrm{C}(X)$.*

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$\widetilde{X} \simeq \widetilde{\mathbb{R}}$

Theorem (Dales et. al. 2016)

Let $\mathbb{I} = [0,1]$, and L an uncountable, second countable metrizable locally compact space (e.g. an uncountable compact metrizable spae). Then $\mathrm{M}(K)$ is isometrically lattice isomorphic to $\mathrm{M}(\mathbb{I})$.

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Corollary

Let K be an uncountable, second countable metrizable locally compact space. Then $\widetilde{K} \simeq \widetilde{\mathbb{I}}$.

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Theorem (vdWalt 2025)

Let X be a metrizable realcompact space. Assume that there exists an increasing sequence $\{K_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_X$ so that

- *$\{K_n\}_{n\in\mathbb{N}}$ is cofinal in $\mathfrak{K}_X$;*
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If there are no measurable cardinals, every metric space is realcompact.

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- $\mathrm{M}_c(X) = \varinjlim \mathrm{M}(K_n) \cong \varinjlim \mathrm{M}(\mathrm{J_n}) = \mathrm{M}_c(\mathbb{R})$
- $\mathrm{C}(\widetilde{X}) = \mathrm{M}_c(X)^{\sim} \cong \mathrm{M}_c(\mathbb{R})^{\sim} = \mathrm{C}(\widetilde{\mathbb{R}}) \Rightarrow \widetilde{X} \simeq \widetilde{\mathbb{R}}$

The End

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