Markov operators on order unit spaces

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Motivation

Let Ω_1 and Ω_2 be non-empty compact Hausdorff spaces. Denote

$$\mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2)) := \{T \colon \mathrm{C}(\Omega_1) \to \mathrm{C}(\Omega_2); \quad T \text{ linear and positive}, T(\mathbb{1}_{\Omega_1}) = \mathbb{1}_{\Omega_2}\}.$$

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Theorem (Phelps/Ellis, 1963).

Let $T \in \mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$. The following are equivalent:

(i) T is an extreme point of $\mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$.

(ii) T is a Riesz homomorphism.

(iii) T is an algebra homomorphism.

(iv) T' maps extreme points of $\mathcal{M}(\mathrm{C}(\Omega_2), \mathbb{R})$ to extreme points of $\mathcal{M}(\mathrm{C}(\Omega_1), \mathbb{R})$.

Functional representation (Kadison, 1951)

Let X be an order unit space with order unit u_X. Define the weakly* compact convex set

$$\Sigma_X := \{\varphi\colon X \to \mathbb{R};\ \varphi \text{ linear and positive}, \varphi(u_X) = 1\}$$

and define Λ_X as the set of extreme points of Σ_X.

Functional representation (Kadison, 1951)

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$$\Sigma_X := \{\varphi\colon X \to \mathbb{R};\ \varphi \text{ linear and positive}, \varphi(u_X) = 1\}$$

and define Λ_X as the set of extreme points of Σ_X.
The weak* closure $\overline{\Lambda_X}$ of Λ_X is a compact Hausdorff space (with the weak* topology) and the map

$$\Phi_X\colon X \to \mathrm{C}(\overline{\Lambda_X}), \quad x \mapsto (\varphi \mapsto \varphi(x)),$$

is linear and bipositive.

Functional representation (Kadison, 1951)

Theorem (Kalauch, Lemmens, van Gaans, 2014).
Let X be an order unit space. Then $\Phi_X[X]$ is *order dense* in $\mathrm{C}(\overline{\Lambda_X})$, i.e., for all $f \in \mathrm{C}(\overline{\Lambda_X})$, one has

$$f = \inf\{\Phi_X(x);\; x \in X, \Phi_X(x) \geq f\}.$$

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$$f = \inf\{\Phi_X(x);\ x \in X, \Phi_X(x) \geq f\}.$$

We have $\Phi_X(u_X) = \mathbb{1}_{\overline{\Lambda_X}}$. Hence:

Order unit spaces $\cong$ *Order dense subspaces of some* $\mathrm{C}(\Omega)$ *space that contain the constant functions*

Generalizations of Riesz homomorphisms

For $M \subseteq X$, we denote
$M^{\mathrm{u}} := \{x \in X;\ \forall m \in M \colon x \geq m\},\ M^{\mathrm{l}} := \{x \in X;\ \forall m \in M \colon x \leq m\}.$

Generalizations of Riesz homomorphisms

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Definition.

Let X, Y ordered vector spaces. A linear map $T\colon X \to Y$ is called a

(a) (van Haandel, 1993) *Riesz* homomorphism* if, for every non-empty finite subset F of X, one has

$$T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}},$$

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$$T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}},$$

(b) (Buskes–van Rooij, 1993) *Riesz homomorphism* if, for every $x, y \in X$, one has

$$T[\{x,y\}^{\mathrm{u}}]^{\mathrm{l}} = T[\{x,y\}]^{\mathrm{ul}}.$$

Riesz and Riesz* homomorphisms on order unit spaces

Let X be an order unit space with order unit u_X. Recall that

$$\Sigma_X = \{\varphi\colon X \to \mathbb{R};\ \varphi \text{ linear and positive}, \varphi(u_X) = 1\},$$
$$\Lambda_X = \operatorname{ext} \Sigma_X.$$

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$$\Lambda_X = \operatorname{ext} \Sigma_X.$$

Proposition.

Let X be an order unit space and let $\varphi \in \Sigma_X$.

(a) (Hayes, 1966) $\varphi \in \Lambda_X$ if and only if φ is a Riesz homomorphism.

(b) (van Haandel, 1993) $\varphi \in \overline{\Lambda_X}$ if and only if φ is a Riesz* homomorphism.

Riesz* homomorphisms on spaces of continuous functions

Theorem (van Imhoff, 2018).

Let Ω_1 and Ω_2 be non-empty compact Hausdorff spaces and let X and Y be order dense subspaces of $\mathrm{C}(\Omega_1)$ and $\mathrm{C}(\Omega_2)$, respectively. Let $T\colon X \to Y$ be linear. Then, under some mild conditions on X, the following statements are equivalent:

(i) T is a Riesz* homomorphism

(ii) There exist $w \in \mathrm{C}(\Omega_2)$, $w \geq 0$, and $\alpha\colon \Omega_2 \to \Omega_1$ continuous on $\{t \in \Omega_2;\ w(t) > 0\}$ such that

$$T(x)(t) = w(t)x(\alpha(t)) \quad (x \in X).$$

Motivation

Let Ω_1 and Ω_2 be non-empty compact Hausdorff spaces. Denote

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Theorem (Phelps/Ellis, 1963).

Let $T \in \mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$. The following are equivalent:

- (i) T is an extreme point of $\mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$.
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Markov operators on order unit spaces

Let $(X, u_X), (Y, u_Y)$ be order unit spaces. We denote

$$\mathcal{M}(X, Y) := \{T\colon X \to Y;\ \ T \text{ linear and positive, } T(u_X) = u_Y\}.$$

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Question.

Let $T \in \mathcal{M}(X, Y)$. Are the following statements equivalent?

(i) T is an extreme point of $\mathcal{M}(X, Y)$.

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(iii) $T'[\Lambda_Y] \subseteq \Lambda_X$.

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If $Y = \mathbb{R}$, this is true by the above ($\mathcal{M}(X, \mathbb{R}) = \Sigma_X$).

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Observation.

$T \in \mathcal{M}(X, Y) \Leftrightarrow T'[\Sigma_Y] \subseteq \Sigma_X$.

Markov operators on order unit spaces

Proposition (Kalauch, S., van Gaans, 2021).

Let $T \in \mathcal{M}(X, Y)$. If $T'[\Lambda_Y] \subseteq \Lambda_X$, then T is a Riesz homomorphism.

Markov operators on order unit spaces

Proposition (Kalauch, S., van Gaans, 2021).

Let $T \in \mathcal{M}(X,Y)$. If $T'[\Lambda_Y] \subseteq \Lambda_X$, then T is a Riesz homomorphism.

Example.

The converse is not true: Let

$$X := \{f \in \mathrm{C}([-1,1]);\ f(0) = \tfrac{1}{2}(f(1)+f(-1))\},$$

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$$T\colon X \to Y,\ T(f) = f,$$
$$\delta_0\colon Y \to \mathbb{R},\ \delta_0(f) = f(0).$$

Then T is a Riesz homomorphism and we have

$$\delta_0 \in \Lambda_Y, \text{ but } T'(\delta_0) \in \overline{\Lambda_X} \setminus \Lambda_X.$$

Hence, $T'[\Lambda_Y] \not\subseteq \Lambda_X$.

Markov operators on order unit spaces

Proposition (S., 2025).

Let $T \in \mathcal{M}(X, Y)$. The following are equivalent:

(i) T is a Riesz* homomorphism.

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Proposition (S., 2025)

Let $(X, u_X), (Y, u_Y)$ be order unit spaces and $T \in \mathcal{M}(X, Y)$. If T satisfies $T'[\Lambda_Y] \subseteq \Lambda_X$, then T is an extreme point of $\mathcal{M}(X, Y)$.

Markov operators on order unit spaces

Example.

Let $X = Y = \mathbb{R}^3$ endowed with the cones

$$X_+ := \mathrm{pos}\{(1,0,1)^\top, (-1,0,1)^\top, (0,1,1)^\top, (0,-1,1)^\top\},$$
$$Y_+ := \{(x_1, x_2, x_3) \in X;\ x_1^2 + x_2^2 \le x_3^2,\ x_3 \ge 0\}.$$

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Then $(0,0,1)^\top =: e^{(3)}$ is an order unit for X and Y and we have

$$\Lambda_X = \overline{\Lambda_X} = \{(1,1,1)^\top, (-1,1,1)^\top, (1,-1,1)^\top, (-1,-1,1)^\top\},$$
$$\Lambda_Y = \overline{\Lambda_Y} = \{(v_1,v_2,1) \in \mathbb{R}^3;\ v_1^2 + v_2^2 = 1\}$$

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One calculates that

$$\mathcal{M}(X,Y) = \{A \in \mathbb{R}^{3\times 3};\ -e^{(3)} \leq_Y a^{(i)} \leq_Y e^{(3)}, i \in \{1,2\},\ a^{(3)} = e^{(3)}\}.$$

Markov operators on order unit spaces

It follows that

$$\operatorname{ext}\mathcal{M}(X,Y) = \{A \in \mathbb{R}^{3\times 3};\ a^{(i)} \in \{\pm e^{(3)}, (\lambda_1,\lambda_2,0)^\top;\ (\lambda_1,\lambda_2) \in S^1\},\ a^{(3)} = e^{(3)}\}.$$

Markov operators on order unit spaces

It follows that

$$\mathrm{ext}\mathcal{M}(X,Y)=\{A\in\mathbb{R}^{3\times 3};\ a^{(i)}\in\{\pm e^{(3)},(\lambda_1,\lambda_2,0)^\top;\ (\lambda_1,\lambda_2)\in S^1\},\ a^{(3)}=e^{(3)}\}.$$

Let now

$$A=\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}.$$

Then $A\in\mathrm{ext}\mathcal{M}(X,Y)$, but

$$A'\begin{pmatrix}1\\0\\1\end{pmatrix}=A^\top\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\notin\overline{\Lambda_X}=\Lambda_X.$$

Note that $(1,0,1)^\top\in\overline{\Lambda_Y}=\Lambda_Y$.

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Let $T \in \mathcal{M}(X, Y)$. Are the following statements equivalent?

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The example before also shows that (i) $\implies$ (ii) does not hold in general.

Markov operators on order unit spaces

Proposition (S., 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces. If every Riesz* homomorphism in $\mathcal{M}(X,Y)$ is extreme in $\mathcal{M}(X,Y)$, then $\Lambda_X = \overline{\Lambda_X}$.

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Remarks.

(a) (Kalauch, S., van Gaans, 2021) The fact that $\Lambda_X = \overline{\Lambda_X}$ already implies that every Riesz* homomorphism $T\colon X \to Y$ is a Riesz homomorphism.

(b) There exist order unit spaces X with $\Lambda_X \neq \overline{\Lambda_X}$.

Markov operators on order unit spaces

Proposition (S., 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces. Assume that $\Lambda_X = \overline{\Lambda_X}$. Let $T \in \mathcal{M}(X, Y)$. If T is a Riesz* homomorphism, then T is extreme in $\mathcal{M}(X, Y)$.

Markov operators on order unit spaces

Proposition (S., 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces. Assume that $\Lambda_X = \overline{\Lambda_X}$. Let $T \in \mathcal{M}(X, Y)$. If T is a Riesz* homomorphism, then T is extreme in $\mathcal{M}(X, Y)$.

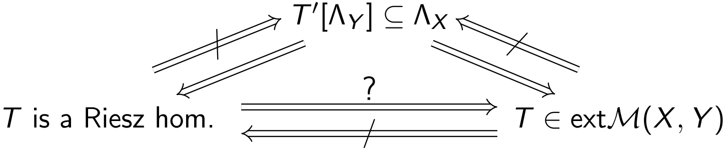
Remarks.

(a) If $\Lambda_X \neq \overline{\Lambda_X}$, then there are examples of Riesz* homomorphisms in $\mathcal{M}(X, Y)$ that are not extreme.

(b) It is still open whether Riesz homomorphisms are extreme in $\mathcal{M}(X, Y)$ if we drop the assumption $\Lambda_X = \overline{\Lambda_X}$.

Markov operators on order unit spaces

Let $(X, u_X), (Y, u_Y)$ be order unit spaces and $T \in \mathcal{M}(X, Y)$.



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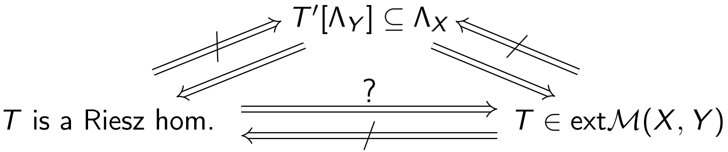
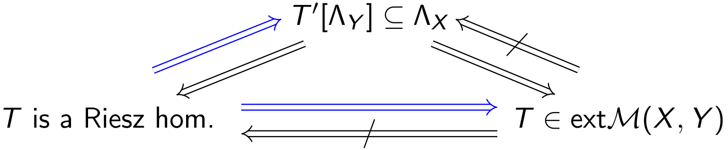


If $\Lambda_X = \overline{\Lambda_X}$, then



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$$\mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2)) := \{T\colon \mathrm{C}(\Omega_1) \to \mathrm{C}(\Omega_2); \quad T \text{ linear and positive}, T(\mathbb{1}_{\Omega_1}) = \mathbb{1}_{\Omega_2}\}.$$

Theorem (Phelps/Ellis, 1963).

Let $T \in \mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$. The following are equivalent:

(i) T is an extreme point of $\mathcal{M}(\mathrm{C}(\Omega_1), \mathrm{C}(\Omega_2))$.

(ii) T is a Riesz homomorphism.

(iii) T is an algebra homomorphism.

(iv) T' maps extreme points of $\mathcal{M}(\mathrm{C}(\Omega_2), \mathbb{R})$ to extreme points of $\mathcal{M}(\mathrm{C}(\Omega_1), \mathbb{R})$.

Motivation

Let A and B be ordered algebras with multiplicative units e_A and e_B. Denote

$$\mathcal{M}(A,B) := \{T\colon A \to B;\ T \text{ linear and positive}, T(e_A) = e_B\}.$$

Theorem (van Putten, 1980).

Let A, B be Archimedean f-algebras and $T \in \mathcal{M}(A,B)$. The following are equivalent:

(i) T is an extreme point of $\mathcal{M}(A,B)$.

(ii) T is a Riesz homomorphism.

(iii) T is an algebra homomorphism.

Markov operators on ordered algebras

Proposition (S., 2025).

Let A, B be Archimedean f-algebras with units $e_A, e_B > 0$, $X \subseteq A$, $Y \subseteq B$ order dense subalgebras, and $T \in \mathcal{M}(X, Y)$.

(a) If T is a Riesz* homomorphism, then T is an algebra homomorphism.

(b) If T is an algebra homomorphism, then T is an extreme point of $\mathcal{M}(X, Y)$.

Markov operators on ordered algebras

Proposition (S., 2025).

Let A, B be Archimedean f-algebras with units $e_A, e_B > 0$, $X \subseteq A$, $Y \subseteq B$ order dense subalgebras, and $T \in \mathcal{M}(X, Y)$.

(a) If T is a Riesz* homomorphism, then T is an algebra homomorphism.

(b) If T is an algebra homomorphism, then T is an extreme point of $\mathcal{M}(X, Y)$.

Corollary (S, 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces such that $\Phi_X[X]$ and $\Phi_Y[Y]$ are *subalgebras* of $\mathrm{C}(\overline{\Lambda_X})$ and $\mathrm{C}(\overline{\Lambda_Y})$.
Then every Riesz* homomorphism $T\colon X \to Y$ is a Riesz homomorphism.

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Proposition (S., 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces such that $\Phi_X[X]$ and $\Phi_Y[Y]$ are *subalgebras* of $\mathrm{C}(\overline{\Lambda_X})$ and $\mathrm{C}(\overline{\Lambda_Y})$.
Let $T\colon X \to Y$ be linear.

(a) If T is a positive algebra homomorphism, then T is a Riesz* homomorphism.

(b) If T is an extreme point of $\mathcal{M}(X, Y)$, then T is an algebra homomorphism.

Markov operators on ordered algebras

Theorem (S., 2025).

Let $(X, u_X), (Y, u_Y)$ be order unit spaces such that $\Phi_X[X]$ and $\Phi_Y[Y]$ are *subalgebras* of $\mathrm{C}(\overline{\Lambda_X})$ and $\mathrm{C}(\overline{\Lambda_Y})$.
Let $T \in \mathcal{M}(X, Y)$. Then the following are equivalent:

(i) T is extreme in $\mathcal{M}(X, Y)$.

(ii) T is an algebra homomorphism.

(iii) T is a Riesz homomorphism.

(iv) T is a Riesz* homomorphism.

(v) $T'[\Lambda_Y] \subseteq \Lambda_X$.

References I

[1] G. Buskes and A.C.M. van Rooij. The vector lattice cover of certain partially ordered groups. J. Austral. Math. Soc. (Series A) **54**, 352-367 (1993).

[2] A.J. Ellis. Extreme positive operators. Quart. J. Math. Oxford Ser. (2) **15**, 342-344 (1964).

[3] A. Hayes. Indecomposable positive additive functionals. J. London Math. Soc. **41**, 318-322 (1966).

[4] A. Kalauch, B. Lemmens, and O. van Gaans. Riesz completions, functional representations, and anti-lattices. Positivity **18**(1), 201-218 (2014).

[5] A. Kalauch, J. Stennder, and O. van Gaans. Operators in pre-Riesz spaces: moduli and homomorphisms. Positivity **25**(5), 2099-2136 (2021).

[6] R.R. Phelps. Extreme positive operators and homomorphisms. Trans. Amer. Math. Soc. **108**, 265–274 (1963).

[7] M. van Haandel. Completions in Riesz Space Theory. PhD thesis, University of Nijmegen (1993).

References II

[8] H. van Imhoff. Riesz* homomorphisms on pre-Riesz spaces consisting of continuous functions. Positivity **22**(2), 425-447 (2018).

[9] B. van Putten. Disjunctive Linear Operators and Partial Multiplications in Riesz Spaces. PhD Thesis. Wageningen University and Research (1980).