Lattice uniformities

Kevin Abela

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For a subset $U \subseteq X \times X$, let $U^{-1} = \{(y, x) : (x, y) \in U\}$. A set satisfying $U = U^{-1}$ is said to be *symmetric* and the diagonal of the product $X \times X$ will be denoted by $\Delta(X)$. Also for sets $U, V \subseteq X \times X$, let $U \circ V$ be the composition $\{(x, z) : (x, y) \in U \text{ and } (y, z) \in V\}$.

A symmetric subset U of $X \times X$ is said to be an *entourage of the diagonal* and the entourages of the diagonal are denoted by $\mathscr{D}_X$.

Definition 1

A *uniformity* on a set X is a subfamily $\mathcal{U}$ of $\mathscr{D}_X$ such that

1. if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;
2. if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
3. if $U \in \mathcal{U}$ and $U \subseteq V \in \mathscr{D}_X$ then $V \in \mathcal{U}$.

For a uniformity $\mathcal{U}$, the pair $(X, \mathcal{U})$ is said to be a uniform space. For a uniform space $(X, \mathcal{U})$ and $U \in \mathcal{U}$, let $U(x) = \{y \in X : (x, y) \in U\}$.

Theorem 2

Let $(X,\mathcal{U})$ be a uniform space. The family $\mathscr{T}_{\mathcal{U}} = \{G \subseteq X :$ for every $x \in G$ there exists a $V \in \mathcal{U}$ such that $V(x) \subseteq G\}$ is a topology on the set X.

The topology $\mathscr{T}_{\mathcal{U}}$ induced in Theorem 2 is known as the **topology induced by the uniformity $\mathcal{U}$**.

Definition 3

A subfamily $\mathcal{X}$ of a uniformity $\mathcal{U}$ is a **base** for a uniformity $\mathcal{U}$ if and only if for every $U \in \mathcal{U}$, there exists $B \in \mathcal{X}$ such that $B \subseteq U$.

Definition 4

A subfamily $\mathcal{Y} \subseteq \mathcal{U}$ is a **subbase** for $\mathcal{U}$ if and only if the family of finite intersections of elements of $\mathcal{Y}$ is a base for $\mathcal{U}$.

Definition 5

A function $f : X \to Y$ from a uniform space $(X, \mathcal{U})$ to a uniform space $(Y, \mathcal{V})$ is said to be uniformly continuous if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $f_2(U) \subseteq V$.i.e for $(x, y) \in U \implies (f(x), f(y)) \in V$.

Lemma 6

Let $f : X \to Y$ be a map from a set X to a uniform space $(Y, \mathcal{U})$. The set $\{f_2^{-1}(U) : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}_f$ on X. Furthermore, $\mathcal{U}_f$ is the smallest uniformity for which f is uniformly continuous.

Lemma 7

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces. The sets $\{((x_1,y_1),(x_2,y_2)) : U \in \mathcal{U}, V \in \mathcal{V}, (x_1,x_2) \in U \text{ and } (y_1,y_2) \in V\}$ is a base for a product uniformity on $X \times Y$. Furthermore, this uniformity induces on $X \times Y$ the product topology $\mathscr{T}_{\mathcal{U}} \times \mathscr{T}_{\mathcal{V}}$.

Theorem 8

Let $(X,\mathcal{U})$ be a uniform space and ρ a pseudo-metric for X. Then, ρ is uniformly continuous on $X \times X$ relative to the product uniformity if and only if the set $U_r = \{(x,y) : \rho(x,y) < r\}$ is an element of $\mathcal{U}$ for every $r > 0$.

Definition 9

Let $(X,\mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ is said to be pseudo-metrizable if there exists a pseudo-metric ρ such that $\mathcal{U} = \mathcal{U}_\rho$.

Theorem 10 (Metrization Theorem)

A uniform space $(X,\mathcal{U})$ is pseudo-metrizable if and only if $\mathcal{U}$ has a countable base.

Theorem 11

Every uniformity on X can be generated by a family of uniformly continuous pseudo-metrics on $X \times X$.

Theorem 12

A topology $\mathscr{T}$ for a set X is the uniform topology for some uniformity for X if and only if the topological space $(X, \mathscr{T})$ is completely regular.

Proposition 13

Let $(X,\mathcal{U})$ be a uniform space. Then, $\mathscr{T}_{\mathcal{U}}$ is Tychanoff if and only if $\bigcap_{U\in\mathcal{U}} U = \Delta(X)$.

Let L be a lattice. A *lattice uniformity* is a uniformity on a lattice making the lattice operations $\vee$ and $\wedge$ uniformly continuous. A uniform lattice is a lattice endowed with a lattice uniformity.

Proposition 14 ([2], H. Weber, 1991)

Let $\mathcal{U}$ be a uniformity on a lattice L. Then the following statements are equivalent :

1. (i) $\mathcal{U}$ is a lattice uniformity;
2. (ii) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ satisfying $V \vee V \subseteq U$ and $V \wedge V \subseteq U$;
3. (iii) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ satisfying $V \vee \Delta(L) \subseteq U$ and $V \wedge \Delta(L) \subseteq U$.

Definition 15

A topology $\mathscr{T}$ is called locally order convex it $\mathscr{T}$ has a base of order convex open sets.

Proposition 16 (H. Weber)

Let $\mathcal{U}$ be a lattice uniformity on L and $\mathscr{T}_{\mathcal{U}}$ the induced topology. Then $(L, \mathscr{T}_{\mathcal{U}})$ is a locally order convex topological lattice. Moreover, every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ such that $V(x)$ is order convex for every $x \in L$.

Proposition 17

Every Hausdorff uniform lattice $(L, \mathcal{U})$ is a sublattice and a dense subspace of a complete (as a uniform space) Hausdorff uniform lattice $(\tilde{L}, \tilde{\mathcal{U}})$.

As uniformities are generated by pseudo-metrics, lattice uniformities are also generated by a family of pseudo-metrics.

Proposition 18 (H. Weber)

Let $(L,\mathcal{U})$ be a uniform lattice and $q > 1$. Then $\mathcal{U}$ is generated by a family of pseudo-metrics P satisfying

$$\rho(x \vee z, y \vee z) \leq \rho(x,y) \quad \text{and} \quad \rho(x \wedge z, y \wedge z) \leq q \cdot \rho(x,y)$$

for every $x, y, z \in L$ and $\rho \in D$.

In Proposition 18, $q \neq 1$ unless L is a distributive lattice.

Remark 19

Recall if $f : L \to (L, \mathcal{U})$ is a map and for every $U \in \mathcal{U}$, $U_f = \{(x, y) \in L^2 : (f(x), f(y)) \in U\}$. The collection $\{U_f : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}_f$.

Theorem 20 (H. Weber)

If $(L, \mathcal{U})$ is a uniform lattice and $f : L \to L$ is a lattice homomorphism. Then, $\mathcal{U}_f$ is a lattice uniformity.

Theorem 21 (H. Weber)

If $(L, \mathcal{U})$ is a uniform lattice and F is a collection of lattice homomorphisms. Then $\mathcal{U}_F$ is a lattice uniformity.

For a lattice L, consider the maps $f_{(a,b)}(x) = (x \wedge b) \vee a$ and let $g_{(a,b)}(x) = (x \vee a) \wedge b$ for $a, b, x \in L$.

Proposition 22

For a lattice L the following statements are equivalent:

1. (i) L is distributive.
2. (ii) $f_{(a,b)}$ is a lattice homomorphism for all $a, b \in L$.
3. (iii) $g_{(a,b)}$ is a lattice homomorphism for all $a, b \in L$.

If L is distributive, then $f_{(a,b)} = f_{(a,a \vee b)} = g_{(a,a \vee b)}$ and $g_{(a,b)} = g_{(a \wedge b,b)} = f_{(a \wedge b,b)}$ for every $a, b \in L$.

Lemma 23

Let $(L,\mathcal{U})$ be a uniform lattice. Then,

1. $\mathcal{U}_{(a,b)} \subseteq \mathcal{U}$;
2. $\mathcal{U}_{(a,b)} \cap [a,b]^2 = \mathcal{U} \cap [a,b]^2$.
3. *If $\mathcal{V}$ is a lattice uniformity such that $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{V}_{(a,b)} \subseteq \mathcal{U}_{(a,b)}$.*

Let $J(L) := \{(a,b) \in L^2 : a \leq b\}$. Then for any $J \subseteq J(L)$, from Theorem 21, one can consider the lattice uniformity $\mathcal{U}_J$ having as subbase the union of the collection of lattice uniformities $\{\mathcal{U}_{(a,b)} : (a,b) \in J\}$.

Proposition 24

Let J and J' be non-empty subsets of $J(L)$.

1. $\mathcal{V}_J \subseteq \mathcal{U}_J \subseteq \mathcal{U}$ for every lattice uniformity $\mathcal{V}$ coarser than $\mathcal{U}$.
2. If for every $(a, b) \in J$ there exists $(a', b') \in J'$ satisfying $a' \leq a \leq b \leq b'$, then $\mathcal{U}_J \subseteq \mathcal{U}_{J'}$ and $(\mathcal{U}_J)_{J'} = (\mathcal{U}_{J'})_J = \mathcal{U}_J$.
3. $\mathcal{U}_J$ is the weakest lattice uniformity which agrees with $\mathcal{U}$ on $[a, b]$ for all $(a, b) \in J$.
4. The topology $\mathscr{T}_{\mathcal{U}_J}$ is the weakest lattice topology which agrees with $\mathscr{T}_{\mathcal{U}}$ on $[a, b]$ for all $(a, b) \in J$.

Let $\mathcal{U}^* = \mathcal{U}_{J(L)}$. Then Proposition 24 above leads to the following corollary

Corollary 25

1. $(\mathcal{U}^*)^* = \mathcal{U}^*$.
2. $\mathcal{U}^*$ *is the weakest lattice uniformity which agrees with* $\mathcal{U}$ *on any order bounded subset of* L.
3. $\mathscr{T}_{\mathcal{U}^*}$*-topology is the weakest lattice topology which agrees with* $\mathscr{T}_{\mathcal{U}}$ *on any order bounded subset of* L.

In the Corollary above, it is deduced that $\mathcal{U}^*$ is the weakest lattice uniformity which coincides with $\mathcal{U}$ on every order bounded subset of L and, $\mathscr{T}_{\mathcal{U}^*}$ is the weakest lattice topology which agrees with $\mathscr{T}_{\mathcal{U}}$. However, there are examples showing that this is not necessarily true for general uniformities.

Proposition 26

Let S be a sublattice of L and $I = J(S) = \{(a,b) \in S^2 : a \leq b\}$. Then $\mathcal{U}_I$ is Hausdorff iff $\mathcal{U}$ is Hausdorff and $x = \sup_{s \in S} s \wedge x = \inf_{s \in S} s \vee x$ for every $x \in L$.

Corollary 27

$\mathcal{U}$ is Hausdorff iff $\mathcal{U}^*$ is Hausdorff.

Definition 28

A (not necessarily Hausdorff) topology τ on a Riesz space V is said to be locally solid if it is linear and has a base at zero consisting of solid sets. The pair (V, τ) where V is a Riesz space and τ is locally solid is known as a locally solid Riesz space.

Definition 29

Let (V, τ) be a locally solid Riesz space and $A \subseteq V$ a Riesz order ideal. A net $(x_\gamma)_{\gamma \in \Gamma}$ in V unbounded τ-converges to $x \in V$ with respect to A if $(|x_\gamma - x| \wedge a)_{\gamma \in \Gamma}$ τ-converges to 0 for every $a \in A$.

Theorem 30 (Taylor, 2019)

Let (V, τ) be a Hausdorff locally solid Riesz space, $\mathfrak{U}$ its 0-neighbourhood system and A a Riesz order ideal of V. Then the sets

$$\{x \in V : |x| \wedge a \in U\} \; (a \in A_+, U \in \mathfrak{U})$$

form a 0-neighbourhood base for a Hausdorff locally solid topology $\mathfrak{u}_A\tau$ on V.

Let (V, τ) be a Hausdorff locally solid Riesz space. A net $(x_\gamma)_{\gamma \in \Gamma}$ converges to x w.r.t. $\mathfrak{u}_A\tau$ iff $|x_\gamma - x| \wedge a \xrightarrow{\tau} 0$ for any $a \in A_+$.

Proposition 31 ([1],E. Chetcuti, H. Weber, K.A)

Let (V, τ) be a Hausdorff locally solid Riesz space and $\mathfrak{U}$ the neighbourhood base of 0 for τ. For any $s, t \in V$ such that $s \leq t$, let

$$U_{(s,t)} = \{(x, y) \in V^2 : f_{(s,t)}(x) - f_{(s,t)}(y) \in U\}.$$

Then $\{U_{(s,t)} : U \in \mathfrak{U}\}$ is a base for a lattice uniformity.

Corollary 32

Let (V, τ) be a Hausdorff locally solid Riesz space and A a Riesz solid subspace of V. If $J = \{(a, b) \in V^2 : b - a \in A_+\}$ and $\mathfrak{U}$ is the neighbourhood base of 0 for τ, then the collection $\{U_{(s,t)} : U \in \mathfrak{U}$ where $(s, t) \in J\}$ is a subbase for the uniformity $\mathcal{U}_J$.

Theorem 33 ([1],E. Chetcuti, H. Weber, K.A)

Let (V, τ) be a Hausdorff locally solid Riesz space and A a Riesz solid subspace of V. Let $\mathcal{U}_J$ be the uniformity induced by $J = \{(a, b) \in V^2 : b - a \in A_+\}$. Then, $\mathscr{T}_{\mathcal{U}_J} = \mathfrak{u}_A\tau$. In particular, $\mathscr{T}_{\mathcal{U}^*} = \mathfrak{u}\tau$.

Let $D(L)$ be the set of pseudo-metrics ρ on L satisfying

$$\rho(x \vee z, y \vee z) \leq \rho(x, y) \quad \text{and} \quad \rho(x \wedge z, y \wedge z) \leq \rho(x, y)$$

and for $J \subseteq J(L)$, $(a, b) \in J$ and $\rho \in D(L)$ let $\rho_{(a,b)}$ denote the pseudo-metric on L defined by $\rho_{(a,b)}(x, y) = \rho(f_{(a,b)}(x), f_{(a,b)}(y))$.

Proposition 34

Let $\mathcal{U}$ be a lattice uniformity on a distributive lattice L. For $J \subseteq J(L)$, the lattice uniformity $\mathcal{U}_J$ is generated by the family of pseudo-metrics $\{\rho_{(a,b)} : \rho \in D_{\mathcal{U}} \text{ and } (a, b) \in J\}$.

Let $(L,\mathcal{U})$ be a uniform lattice and $J \subseteq J(L)$. Let $\tilde{J} = \{(a,b) : a \leq b$ and there exists $(a_\alpha, b_\alpha) \in J$ such that $a_\alpha \to a$ and $b_\alpha \to b$ w.r.t $\mathscr{T}_{\mathcal{U}}\}$.

Proposition 35

Let $\emptyset \neq J \subseteq J(L)$. Then $\mathcal{U}_J = \mathcal{U}_{\tilde{J}}$.

Corollary 36

Let S be a dense sublattice of $(L,\mathcal{U})$ and $\mathcal{V} = \mathcal{U}|_S$. Then $\mathcal{U}^*|_S = \mathcal{V}^*$.

Theorem 37

Let S be a dense sublattice of $(L,\mathcal{U})$ and $\mathcal{V} = \mathcal{U}|_S$.

1. *Then $\mathcal{U}^* = \mathcal{U}$ iff $\mathcal{V}^* = \mathcal{V}$.*
2. *If $\mathscr{T}_{\mathcal{U}^*} = \mathscr{T}_{\mathcal{U}}$, then $\mathscr{T}_{\mathcal{V}^*} = \mathscr{T}_{\mathcal{V}}$.*

Theorem 38 ([1],E. Chetcuti, H. Weber, K.A)

Let H be a dense Riesz subspace of a locally solid Riesz space (V,τ) and let $\sigma = \tau|_H$. Then $\mathfrak{u}\tau = \tau$ iff $\mathfrak{u}\sigma = \sigma$.

[1] Kevin Abela, Emmanuel Chetcuti, and Hans Weber, *Lattice uniformities inducing unbounded convergence*, J. Math. Anal. Appl. **523** (2023), no. 1, Paper No. 126994, 18.

[2] Hans Weber, *Uniform lattices. I. A generalization of topological Riesz spaces and topological Boolean rings*, Ann. Mat. Pura Appl. (4) **160** (1991), 347–370 (1992). MR 1163215