Bandwise functions in Dedekind complete Φ-algebras

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Dedekind complete Φ-algebras

Throughout this talk, E will be a (real or complex) Dedekind complete Φ-algebra.

Definition:

A real Φ-algebra is a real vector lattice F with an associative multiplication which has a unit and that satisfies the usual algebra properties as well as the following:

- *if* $x, y \geq 0$ *then* $xy \geq 0$, *and*
- *if* $x \wedge y = 0$, *and* $z \geq 0$, *then* $(xz) \wedge y = (zx) \wedge y = 0$.

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If F is Dedekind complete, it can be complexified to a complex Φ-algebra $E = F + iF$, where the modulus is given by

$$|x + iy| = \sup\{(\cos\theta)x + (\sin\theta)y : \theta \in [0, 2\pi]_{\mathbb{R}}\}.$$

Order convergence

Notation:

For $\mathcal{E} \subseteq E$, we write $\mathcal{E} \searrow 0$ to mean $\mathcal{E}$ has infimum 0, and $\mathcal{E} \downarrow 0$ if $\mathcal{E}$ is also downwards directed

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Definition:

A net $(x_\alpha)_\alpha$ ***converges (in order)*** *to x in E if there exists an $\mathcal{E} \searrow 0$ such that for every $\varepsilon \in \mathcal{E}$ there exists an α_0 such that $|x_\alpha - x| \leq \varepsilon$ holds for all $\alpha \geq \alpha_0$.*

Order continuity

Definition:

A function $f : \mathrm{dom}(f) \to E$ *is* ***(order) continuous at*** x *if whenever* $x_\alpha \to x$ *in* $\mathrm{dom}(f)$*, then* $f(x_\alpha) \to f(x)$ *in* E*.*

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Proposition:

A function $f : \mathrm{dom}(f) \to E$ *is continuous at* x *if and only if for all* $\Delta \downarrow 0$ *there exists an* $\mathcal{E} \searrow 0$ *such that for every* $\varepsilon \in \mathcal{E}$ *there exists a* $\delta \in \Delta$ *satisfying*

$$y \in \mathrm{dom}(f) \text{ and } |y - x| \leq \delta \implies |f(y) - f(x)| \leq \varepsilon.$$

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Definition:

Let $c, r \in E$ and $r \geq 0$. We define

$$\overline{N}(c,r) := \{z \in E : |z - c| \leq r\}, \text{ and}$$
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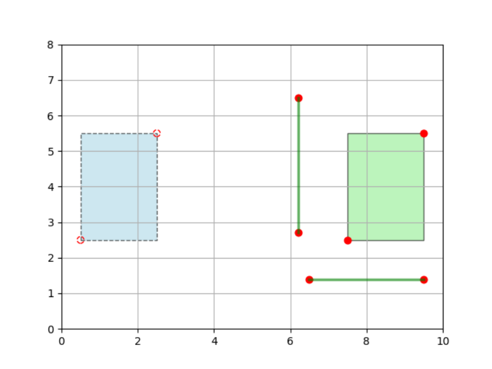
If E *is real, we also use interval notation: For* $a \leq b$*, define*

$$[a,b] := \{x \in E : a \leq x \leq b\}, \text{ and}$$
$$(a,b) := \{x \in E : a \ll x \ll b\} \quad (a \ll b).$$

Example: Intervals

Example:

In $\mathbb{R}^2$, the open intervals are the interiors of rectangles. The closed intervals are rectangles, and horizontal and vertical line segments.



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2. *for every* $\Delta \downarrow 0$ *there is an* $\mathcal{E} \searrow 0$ *such that for all* $\varepsilon \in \mathcal{E}$*, there is a* $\delta \in \Delta$ *such that for all* $z \in N(c, r)$ *(resp.* $z \in \operatorname{dom}(f)$*),* $|z - c| \leq \delta \implies |f(z) - f(c) - (z - c)f'(c)| \leq |z - c|\varepsilon.$

Properties of the derivative

Proposition:

Consider $f\colon \operatorname{dom}(f) \to E$ and $g\colon \operatorname{dom}(g) \to E$.

1. *If f and g are (super) differentiable at c, then $f+g$ is (super) differentiable at c with $(f+g)'(c) = f'(c) + g'(c)$.*
2. *If f and g are (super) differentiable at c, then fg is (super) differentiable at c with $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.*

Proposition:

If $f : \operatorname{dom}(f) \to E$ is super differentiable at c then f is continuous at c.

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For any $y \in (x - r, x + r)$, $|x_n - y_n| < \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *so* $y_n \to \frac{1}{2}$ *if and only if* $x_n \to \frac{1}{2}$.

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Therefore, f *is constant on* $(x - r, x + r)$ *and* $f'(x) = 0$.

Bandwise functions

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A function $f\colon \operatorname{dom}(f) \to E$ *is* ***bandwise*** *if for any* $x, y \in \operatorname{dom}(f)$ *and any band projection* $\mathbb{P}$, $\mathbb{P}x = \mathbb{P}y$ *implies* $\mathbb{P}f(x) = \mathbb{P}f(y)$.

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A function $f\colon \operatorname{dom}(f) \to \mathbb{R}^2$ *is bandwise if and only if* $f(x, y) = (g(x), h(y))$ *for some* $g, h\colon \mathbb{R} \to \mathbb{R}$.

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Proposition:

If $f\colon N(c, r) \to E$ is super differentiable, then f is bandwise.

Band decompositions of E

For any $s, t \in \mathbb{R}$, we have that exactly one of the following holds:

$$s < t, \quad s > t, \quad \text{or} \quad s = t.$$

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$$B_{x<y} := B_{(y-x)^+}, \quad B_{x\leq y} := B^d_{y<x}, \quad \text{and} \quad B_{x=y} := B_{x\leq y} \cap B_{y\leq x}.$$

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Proposition:

For any $x, y \in E$,

$$E = B_{x<y} \oplus B_{x\geq y} = B_{x<y} \oplus B_{x>y} \oplus B_{x=y}.$$

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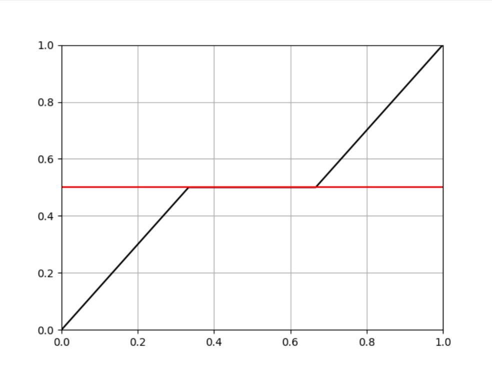

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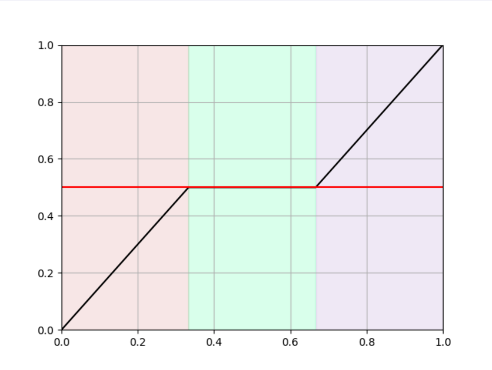

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Let $f : [a, b] \to E$ be continuous and bandwise, and let $y \in [f(a) \wedge f(b), f(a) \vee f(b)]$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.

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The Extreme Value Theorem

Let $f : [a, b] \to E$ be continuous and bandwise. Then there exist $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

The Mean Value Theorem

Let $f : [a, b] \to E$ be continuous on $[a, b]$ and super differentiable on (a, b). Then there exists an $x_0 \in (a, b)$ such that

$$(b - a)f'(x_0) = f(b) - f(a).$$

Classical theorems (cont.)

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The Riemann integral in E

Consider a bounded function $f : [a, b] \to E$, where $[a, b] \subseteq E$.

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For a given interval $I \subseteq [a, b]$, we define

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Definition:

A ***partition*** *of $[a,b]$ is a totally ordered, finite subset of $[a,b]$, say $P = \{a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$. We define the* ***lower*** *and* ***upper sums*** *of f with respect to P by*

$$L(f,P) := \sum_{i=1}^{n} m_{[x_{i-1},x_i]}(x_i - x_{i-1})$$

$$U(f,P) := \sum_{i=1}^{n} M_{[x_{i-1},x_i]}(x_i - x_{i-1}).$$

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The ***lower*** *and* ***upper integrals*** *of f are defined by*

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We say f is ***(Riemann) integrable*** *on $[a,b]$ if $U(f) = L(f)$ and in this case, we write*

$$\int_a^b f(x)dx := U(f) = L(f).$$

Differences from the classical theory

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So, $L(f,P) \not\leq U(f,Q)$.

- $P \cup Q$ is not a partition.
- There is no partition R such that $L(f,P), L(f,Q) \leq L(f,R)$ and $U(f,P), U(f,Q) \geq U(f,R)$.

Partitions and bandwise functions

- If f is bandwise with partitions P and Q, there is a partition R such that

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- But we cannot take $R = P \cup Q$.
- If $f : [0,b] \to E$ is bandwise, P a partition of $[0,b]$ and $\mathbb{P}$ is a band projection, then

$$\mathbb{P}L(f,P) = L(f,\mathbb{P}P) \text{ and } \mathbb{P}U(f,P) = U(f,\mathbb{P}P).$$

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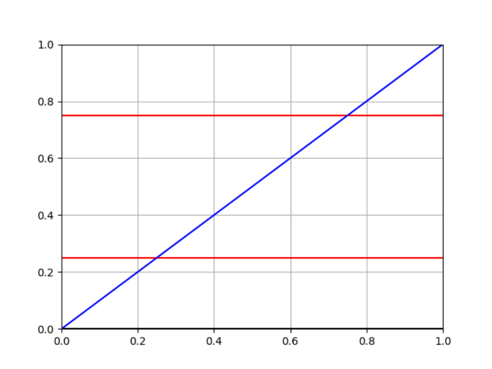
Consider $[0,1]$ *in* $\mathbb{R}^{[0,1]}$.



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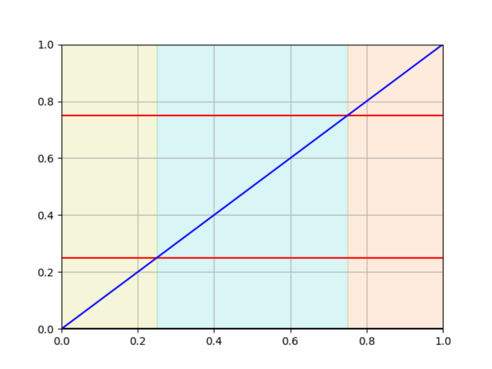

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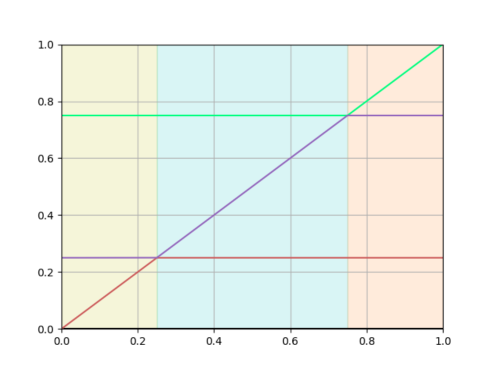

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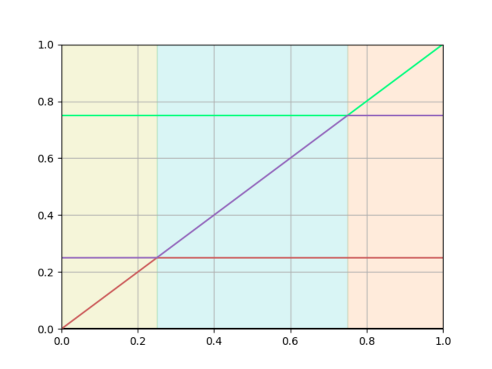

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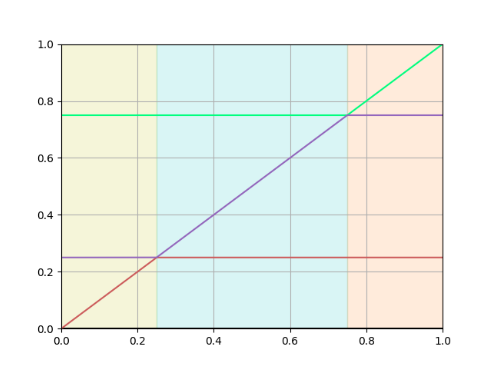

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$$\mathbb{P}(L(f,P)) = L(f,\mathbb{P}(P)) \leq L(f,\mathbb{P}(R)) = \mathbb{P}(L(f,R)).$$

Integrability condition for bandwise functions

For a bandwise function f, $U(f, P) \downarrow U(f)$ and $L(f, P) \uparrow L(f)$.

Proposition:

Let $f : [a, b] \to E$ be bandwise. Then f is integrable if and only if there exists an $\mathcal{E} \searrow 0$ such that for every $\varepsilon \in \mathcal{E}$ there exists a partition P such that $U(f, P) - L(f, P) \leq \varepsilon$.

Properties of the integral

Proposition:

Let $f, g : [a, b] \to E$ be bounded and bandwise, and let $c \in E$. If f and g are integrable, then

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3. *if $f \le g$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$, and*
4. *$|f|$ is integrable and $\int_a^b f(x)dx \le \int_a^b |f(x)|dx$.*

Uniform continuity

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A function $f : \mathrm{dom}(f) \to E$ *is* ***uniformly continuous*** *if whenever* $x_\alpha - y_\alpha \to 0$ *in* $\mathrm{dom}(f)$ *then* $f(x_\alpha) - f(y_\alpha) \to 0$.

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Proposition:

Let $f : [a, b] \to E$ *be uniformly continuous and bandwise. Then* f *is integrable on* $[a, b]$.

A Fundamental Theorem of Calculus

Proposition:

Let $a \ll b$. Suppose $f : [a, b] \to E$ is uniformly continuous and bandwise. Then, $F(x) := \int_a^x f(t)dt$ is uniformly continuous on $[a, b]$ and super differentiable on (a, b) with $F' = f$. Moreover, for any function $G : [a, b] \to E$ the following are equivalent:

1. *G is uniformly continuous on $[a, b]$ and super differentiable on (a, b) with $G' = f$, and*
2. *$\int_x^y f(t)dt = G(y) - G(x)$ for any $x, y \in [a, b]$.*

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The End

Thank you for your attention.