Positivity of boundary delay systems

Positivity XII

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Overview

Semigroup approach to delay equations

Abstract boundary systems in a nutshell

Abstract boundary delay systems

Example: Delayed transport in metric graphs

Semigroup approach to delay equations

Semigroup approach to delay equations

Let X be a Banach lattice and $A\colon D(A) \subset X \to X$ linear operator.

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = f \in X. \end{cases}$$

- (ACP) is *well posed* with solution $u(t) = T(t)f \iff A$ generates a *C_0-semigroup* $(T(t))_{t\geq 0}$ on X ($T(t) \rightsquigarrow e^{tA}$).

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- $(T(t))_{t\geq 0}$ is positive $\iff R(\lambda, A) \geq 0$ for λ big enough.
- If A generates a positive C_0-semigroup and $B \in \mathcal{L}(X)$ is positive then $A + B$ generates a positive semigroup on X.

Abstract Delay Equation

$$\begin{cases} \dot{u}(t) = Au(t) + \int_{-1}^{0} d\eta(s)u(s+t), \quad t \geq 0, \\ u(0) = f \in X, \\ u(s) = \varphi(s), \ s \in [-1, 0]. \end{cases}$$

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Introducing the *history function* $u_t \colon [-1,0] \to X$, $u_t(s) := u(t+s)$ and the *delay operator* $Pg := \int_{-1}^{0} d\eta(s)g(s)$ we get

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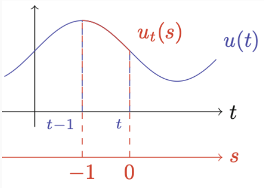

Semigroup approach to delay equations

$$\mathcal{U}\colon t \mapsto \binom{u(t)}{u_t} \in X \times L^p([-1,0],X)$$

$$\mathcal{A}_P := \begin{pmatrix} A & P \\ 0 & \frac{d}{ds} \end{pmatrix}$$

$$D(\mathcal{A}_P) := \left\{ \binom{f}{\varphi} \in D(A) \times W^{1,p}([-1,0],X) :\ \varphi(0) = f \right\}$$

$$(DE) \iff \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) = \binom{f}{\varphi}. \end{cases}$$

[1] A. Bátkai, M.K.F., A. Rhandi, *Positive operator semigroups: From finite to infinite dimensions*, 2017 (Ch. 15)

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Proposition[1]
If A generates a positive C_0-semigroup on X and P is positive then $\mathcal{A}$ generates a positive C_0-semigroup on $X \times L^p([-1,0],X)$.

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By control-theoretical approach á la Staffans-Weiss one can rewrite

$$A_m = (A_{-1} + BG)_{|D(A_m)}$$

and obtain the integral solution

$$u(t) = T(t)f + \underbrace{\int_0^t T_{-1}(t-s)Bu(s)ds}_{=:\Phi_t u} \in X_{-1}.$$

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- The triple operator (A, B, C) is regular and $I_U - F_\tau$ invertible in $L^p([0,\tau], U)$ for some $\tau > 0$.

Theorem

Under these asssumptions (BS) is well-posed.[2] [3]

If the operators $M, R(\lambda, A)$ and D_λ are positive for all $\lambda > s(A)$ and there exists $\lambda_0 > s(A)$ such that $r(MD_{\lambda_0}) < 1$ then the solution semigroup is *positive*.[4] [5]

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[2] M. Adler, M. Bombieri, K.-J. Engel, *On Perturbations of Generators of C_0-Semigroups*, Abstract and Applied Analysis, Article ID 213020, 2014.

[3] S. Hadd, R. Manzo, A. Rhandi, *Unbounded perturbations of the generator domain* Disc. Cont. Dyn. Sys. A, 35 (2015), 703–723.

[4] A. Boulouz, H. Bounit and S. Hadd, Feedback theory approach to positivity and stability of evolution equations, Systems & Control Lett. 161, (2022) 105167

[5] A. Barbieri, K.-J. Engel, On Structured Perturbations of Positive Semigroups. arXiv:2405.18947

Abstract boundary delay systems

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$$(BDS)\begin{cases} \dot{u}(t) = A_m u(t) + Pu_t, & t \geq 0, \\ Gu(t) = Mu(t) + Lu_t, & t \geq 0, \\ u(0) = f,\ u(s) = \varphi(s), & s \in [-1, 0]. \end{cases}$$

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(BDS)\begin{cases} \dot{u}(t) = A_m u(t) + P u_t, & t \geq 0, \\ G u(t) = M u(t) + L u_t, & t \geq 0, \\ u(0) = f,\ u(s) = \varphi(s), & s \in [-1, 0]. \end{cases}
$$

Our path

Take $\mathcal{A}_{P,L} := \begin{pmatrix} A_m & P \\ 0 & \frac{d}{ds} \end{pmatrix}$ with domain consisting of

$$
\binom{f}{\varphi} \in D(A_m) \times W^{1,p}([-1,0], X) : (G - M)f = L\varphi,\ f = \varphi(0)
$$

and follow the steps needed to prove the well-posedness of (BS).

Main Theorem[6]
If the operators $M, L, P, R(\lambda, A)$ and D_λ are positive for all $\lambda > s(A)$ and there exists $\lambda_0 > s(A)$ such that $r(MD_{\lambda_0}) < 1$ then the semigroup $(\mathcal{T}_{P,L}(t))_{t\geq 0}$ is *positive*.

[6] A. Bátkai, M.K.F, A. Rhandi, *Abstract boundary delay systems and application to network flow*, arXiv:2503.08809

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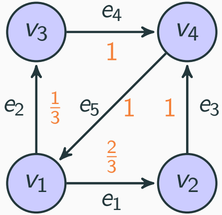

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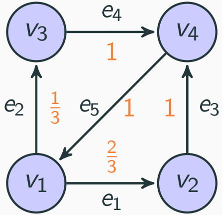

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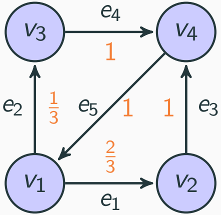

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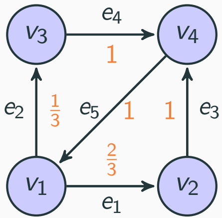

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- on the edges of a finite, connected *metric graph* ($e_j \equiv [0, 1]$, $j = 1, \dots, m$)

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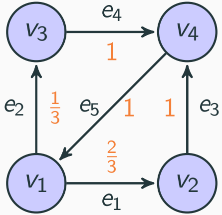

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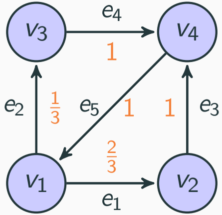

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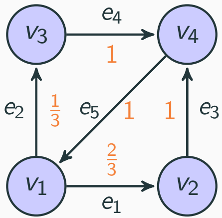

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- in each vertex v_i the incoming material is distributed into the outgoing edges e_j according to weights $0 \leq \omega_{ij} \leq 1$
- the mass is conserved: $\sum \text{inflow} = \sum \text{outflow}$
- the flow is delayed along the edges as well as in the vertices.

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Rewrite as IBVP

$$\begin{cases} \frac{d}{dt} u_j(x,t) = \frac{d}{dx} u_j(x,t), & t \geq 0,\ x \in (0,1), \\ u_j(1,t) = \sum_{k=1}^{m} \mathbb{B}_{jk} u_k(0,t), & t \geq 0, \\ u_j(x,0) = f_j(x), & x \in (0,1). \end{cases}$$

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- $\mathbb{B}$ is the weighted adjacency matrix of the line graph

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- $f(x)$ is the initial mass distribution

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IBVP with delays

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Delay operators

$$P_k g_k := \int_{-1}^{0} d\eta_k(s) g_k(s) \quad \text{and} \quad \ell_k g_k := \int_{-1}^{0} d\eta_k(s) g_k(s)$$

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Corollary
The network system with delays is well-posed. The solutions are positive if the initial functions are positive.

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- (A, B, C) is regular ...
- $$(F_t u)(t) = \begin{cases} \mathbb{B}u(t-1), & \text{if } t \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

  and $I_{\mathbb{C}^m} - F_t$ is invertible for $t < 1$.

Some steps of the proof

- G is surjective and A with $D(A) = \{f : f(1) = 0\}$ generates the left translation semigroup $T(\cdot)$ on X.
- Dirichlet operator: $(D_\lambda a)(x) = e^{\lambda(x-1)}a$, $a \in \mathbb{C}^m$, $x \in [0,1]$
- (A, B, C) is regular ...
- $$(F_t u)(t) = \begin{cases} \mathbb{B}u(t-1), & \text{if } t \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

  and $I_{\mathbb{C}^m} - F_t$ is invertible for $t < 1$.
- Finaly, we obtain positivity since $MD_\lambda = e^{-\lambda}\mathbb{B}$ so $r(MD_\lambda) < 1$ for all $\lambda > 0$. $\square$

Thank you for your attention!