From order unit spaces to Jordan-Banach algebras

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Positivity XII

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Order unit spaces

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- V_+ is *Archimedean*, that is, $\{\lambda x \colon \lambda \geqslant 0\}$ has an upper bound in V only if $x \leqslant 0$;
- there is $v \in V_+$ such that for all $x \in V$ there is $\lambda > 0$ for which $x \leqslant \lambda v$. Such a v is called an *order unit*.

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With respect to $\|\cdot\|_v$:

- ☞ V_+ is closed;
- ☞ interior V_+° of V_+ consists of the order units of V.

Order unit spaces

Example

Let K be a compact and convex subset of a locally convex space. Then

$$A(K) := \{f \colon K \to \mathbb{R} \colon f \text{ is affine and continuous}\}$$

with cone

$$A(K)_+ := \{f \in A(K) \colon f(\omega) \geqslant 0 \text{ for all } \omega \in K\}$$

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and order unit $\mathbf{1}_K$ is an order unit space.

- Note that the order unit norm on $A(K)$ is the maximum norm $\|\cdot\|_\infty$.

Order unit spaces

Theorem (Kadison, 1951)

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Algebraic operations do not interact well with this "observability":

- ☞ scalar multiplication,
- ☞ matrix multiplication (composition of operators).

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- Study algebraic properties of Hermitian matrices (self-adjoint operators) to formulate formal algebraic properties and see what other systems satisfy these axioms.

Jordan-Banach algebras

Definition

A real Banach space A equipped with bilinear product $\circ$ is called a *JB-algebra*, if for all $x, y \in A$ the product satisfies:

- ☞ $x \circ y = y \circ x$ (commutative),
- ☞ $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ (Jordan identity),

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✓ We will assume all JB-algebras in this talk to be unital (unit is denoted by e).

Examples

Prototypical Example

Let A be a unital C^*-algebra. Then the self-adjoint part A_{sa} equipped with the product

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It is clear that this product is commutative, and it is a straightforward verification that the Jordan identity is satisfied.

Examples

To illustrate this, consider the self-adjoint bounded operators $B(H)_{sa}$ on some complex Hilbert space H.

- Sub-multiplicativity:

$$\begin{aligned}\|T\circ S\| = \|\tfrac{1}{2}(TS+ST)\| &\leqslant \tfrac{1}{2}\|TS\| + \tfrac{1}{2}\|ST\| \\ &\leqslant \tfrac{1}{2}\|T\|\|S\| + \tfrac{1}{2}\|S\|\|T\| = \|T\|\|S\|.\end{aligned}$$

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- C^*-algebra property:

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- Monotonicity: For $\xi \in H$ with $\|\xi\| \leqslant 1$,

$$\begin{aligned}\|T\xi\|^2 = \langle T\xi, T\xi\rangle &\leqslant \langle T\xi, T\xi\rangle + \langle S\xi, S\xi\rangle \\ &= \langle (T^2 + S^2)\xi, \xi\rangle \leqslant \|(T^2 + S^2)\xi\|.\end{aligned}$$

Order on JB-algebras

Theorem

- Every JB-algebra A is partially ordered by the cone of squares

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✓ Hence JB-algebras are order unit spaces.

Finite dimensional JB-algebras

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A finite dimensional unital Jordan algebra A over $\mathbb{R}$ is called *Euclidean* if there exists an inner product on A for which Jordan multiplication is symmetric.

That is,

$$\langle x \circ y, z \rangle = \langle y, x \circ z \rangle \qquad \text{(for all } x, y, z \in A).$$

Example

The algebra of $n \times n$ self-adjoint matrices $\mathrm{Herm}_n(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, equipped with the Jordan product

$$M \circ N := \frac{1}{2}(MN + NM)$$

and inner product $\langle M, N \rangle := \mathrm{trace}(M \circ N) = \mathrm{trace}(MN)$.

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Indeed, for $M, N, P \in \mathrm{Herm}_n(\mathbb{F})$, we find

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Finite dimensional JB-algebras

Finite dimensional JB-algebras ↔ Euclidean Jordan algebras

- every Euclidean Jordan algebra can be renormed to be a JB-algebra;
- every finite dimensional JB-algebra can be equipped with an inner product turning it into a Euclidean Jordan algebra.

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- every Euclidean Jordan algebra can be renormed to be a JB-algebra;
- every finite dimensional JB-algebra can be equipped with an inner product turning it into a Euclidean Jordan algebra.

✓ Hence the finite dimensional JB-algebras are precisely the Euclidean Jordan algebras.

Spin factors

Let H be a real Hilbert space and consider $H \times \mathbb{R}$ equipped with the product

$$(x, \lambda) \circ (y, \mu) := (\mu x + \lambda y, \langle x, y \rangle + \lambda \mu),$$

inner product

$$\langle (x, \lambda), (y, \mu) \rangle := \langle x, y \rangle + \lambda \mu,$$

and norm

$$\|(x, \lambda)\| := \sqrt{\langle x, x \rangle} + |\lambda|.$$

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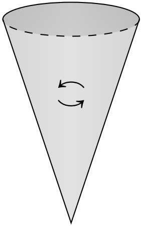
- It is a Hilbert space for the inner product.
- These JB-algebras are called *spin factors*.

General question

Algebra structure from properties of the cone

From which properties of a cone V_+ in an order unit space can we conclude that V is a JB-algebra?



$$\overset{?}{\Longleftrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{nn} \end{bmatrix}$$

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For an inner product $\langle \cdot, \cdot \rangle$ on V, the *dual cone* V_+^* is defined by

$$V_+^* := \{x \in V : \langle y, x \rangle \geqslant 0, \text{ for all } y \in V_+\}.$$

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The cone V_+ is called *self-dual* if $V_+ = V_+^*$.

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- In $A := \mathrm{Herm}_n(\mathbb{F})$, the cone A_+ is
$$\{\text{positive semi-definite matrices}\} = \{B^*B \colon B \in \mathrm{Mat}_n(\mathbb{F})\};$$
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$$M^{-1/2}NM^{-1/2} = M^{-1/2}B^*BM^{-1/2} = (BM^{-1/2})^*BM^{-1/2}.$$

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Since $TM = I$, it follows that A_+ is homogeneous.

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For $M, N \in A_+$, we see that

$$\begin{aligned}\mathrm{trace}(NM) &= \mathrm{trace}(N(\lambda_1 P_1 + \cdots + \lambda_n P_n)) \\ &= \sum_{k=1}^{n} \lambda_k \mathrm{trace}(NP_k) = \sum_{k=1}^{n} \lambda_k \mathrm{trace}(P_k N P_k) \geqslant 0,\end{aligned}$$

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because $P_k N P_k = P_k B^* B P_K = (BP_k)^* BP_k$.

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because $P_k N P_k = P_k B^* B P_K = (BP_k)^* BP_k$.

If $M = QDQ^*$ is such that $\mathrm{trace}(NM) \geqslant 0$ for all $N \in A_+$, then $N := QE_{kk}Q^* \in A_+$ and

$$\lambda_k = \mathrm{trace}(E_{kk} D) = \lambda_k \mathrm{trace}(QE_{kk} QQ^* DQ^*) = \mathrm{trace}(NM) \geqslant 0,$$

so $M \in A_+$. Hence A_+ is self-dual.

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Theorem (Koecher-Vinberg, ∼1960)

Let (V, V_+, v) be a finite dimensional order unit space. If V_+ is symmetric, then V is a Euclidean Jordan algebra with unit v and cone of squares V_+.

Order theoretical characterisation of JB-algebras

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Example

For $\mathrm{Herm}_n(\mathbb{F})$: use that $N \mapsto M^{-1/2}NM^{-1/2}$ is in $\mathrm{Aut}(A_+)$.

$$N \leqslant M \Rightarrow M^{-1/2}NM^{-1/2} \leqslant I \Rightarrow I \leqslant M^{1/2}N^{-1}M^{1/2}$$
$$\Rightarrow M^{-1} \leqslant N^{-1}.$$

Conversely, the existence of a gauge-reversing bijection implies that there is a Jordan algebra structure!

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- Goal: to prove an infinite dimensional characterisation for JB-algebras using gauge-reversing bijections.

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- First step: assume the cone is strictly convex.

Order theoretical characterisation of JB-algebras

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Theorem (v. Imhoff, Lemmens, R., 2017)

Let (V, V_+, v) be a complete order unit space with strictly convex cone. If there exists a gauge-reversing bijection $\Phi \colon V_+^\circ \to V_+^\circ$, then V is a spin factor with cone of squares V_+.

✓ Note that the JB-algebras with strictly convex cones are precisely the spin factors.

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Theorem (Lemmens, R., Wortel, 2025)

Let (V, V_+, v) be a reflexive order unit space. If there exists a gauge-reversing bijection $\Phi\colon V_+^\circ \to V_+^\circ$, then V is a finite order and algebra direct sum of spin factors and Euclidean Jordan algebras with unit v and cone of squares V_+.

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Order theoretical characterisation of JB-algebras

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Theorem (R., Tiersma, 2025)

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✓ Further details of these results will be discussed in the talk by Samuel Tiersma.

Thank you for your attention!