On positive commutators of positive operators

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Let $\mathcal{A}$ be an associative algebra over a field F. An element $c \in \mathcal{A}$ is a **commutator**, if it can be written as

$$c = ab - ba =: [a, b]$$

for some $a, b \in \mathcal{A}$.

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Jaroslav Zemánek's Question

- Let A and B be non-negative matrices with a non-negative commutator $AB - BA$. Is $AB - BA$ a nilpotent matrix?
- Let A and B be positive compact operators on a Banach lattice with a positive commutator $AB - BA$. Is $AB - BA$ quasinilpotent?

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Question

Which $n \times n$ matrices over a field F are commutators?

Theorem (Shoda 1937, Albert, Muckenhoupt 1957)

A matrix over a field is a commutator if and only if it has zero trace.

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Hilbert space case history

- The identity operator on a Hilbert space is not a commutator of bounded operators. (Wintner 1947)
- The unit element of a normed algebra is not a commutator. (Wielandt 1949)
- Operators of the form $K + \lambda I$ with K compact and $\lambda \neq 0$ are not commutators. (Halmos 1963)
- Every compact operator is a commutator (if $\dim \mathcal{H} = \infty$). (Brown, Halmos, Pearcy 1965)

- Separable case: A bounded operator is not a commutator if and only if it is of the form $K + \lambda I$ for some compact operator K and a nonzero scalar λ. (Brown, Pearcy 1965)
- General case: If $\dim \mathcal{H} = \aleph > \aleph_0$, then $\mathcal{B}(\mathcal{H})$ has a the largest ideal $\mathcal{I}_{\aleph}$.

Theorem (Brown, Pearcy 1965)

An operator in $\mathcal{B}(\mathcal{H})$ is not a commutator if and only if it is of the form $K + \lambda I$ for some $K \in \mathcal{I}_{\aleph}$ and some nonzero scalar $\lambda \neq 0$.

If $\mathcal{H}$ is separable, then $\mathcal{J} = \mathcal{K}(\mathcal{H})$ is the largest proper ideal in $\mathcal{B}(\mathcal{H})$.

Banach space history

Theorem (Schneeberger, 1971)

Every compact operator on an infinite-dimensional separable Banach space $L^p(\mu)$ $(1 \le p < \infty)$ is a commutator.

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An operator on a Banach space X is not a commutator if and only if it is of the form $K + \lambda I$ for some compact operator K and a nonzero scalar λ in the following cases.

- ℓ^p $(1 < p < \infty)$ (Apostol 1972) and c_0 (Apostol 1973)
- ℓ^1 (Dosev 2009)

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Theorem (Dosev, Johnson 2010)

An operator on ℓ^∞ is not a commutator if and only if it is of the form $K + \lambda I$ for some strictly singular operator K and some nonzero scalar λ.

Theorem (Dosev, Johnson, Schechtman 2013)

An operator on $L^p[0,1]$ $(1 \leq p < \infty)$ is not a commtuator if and only if it is of the form $K + \lambda I$ where K belongs to the largest ideal in $\mathcal{B}(L^p[0,1])$ and some scalar $\lambda \neq 0$.

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Conjecture

Let X be a Banach space for which we have $X \cong (\sum X)_p$ $(1 \leq p \leq \infty)$ or $p = 0$. Suppose that $\mathcal{B}(X)$ has the largest ideal $\mathcal{J}$. Then a bounded operator on X is not a commutator if and only if it is of the form $K + \lambda I$ for some $K \in \mathcal{J}$ and $\lambda \neq 0$.

Questions and problems

Let A and B be positive operators on a Banach lattice with a positive commutator $C := AB - BA$.

- Determine spectral properties of C (spectral radius, connection with the Jacobson radical, ...).
- Can a positive commutator of positive operators be invertible?
- Which positive operators are commutators of positive operators?
- Determine the dimension of the algebra generated by special positive operators A and B (at least in case for matrices)
- ...

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Hilbert space setting is boring! ($C \geq D \Leftrightarrow C - D$ is positive semi-definite)

$$AB - BA = (AB - BA)^* = B^*A^* - A^*B^* = BA - AB$$

The Hilbert space ℓ^2 is a Banach lattice. Consider the Hilbert space $\mathcal{H} = \bigoplus_{i=1}^{\infty} \ell^2$ ordered coordinatewise. Choose an increasing bounded sequence $0 \leq T_1 \leq T_2 \leq \cdots$ of positive operators on ℓ^2. Then

$$A = \begin{bmatrix} 0 & T_1 & 0 & 0 & \cdots \\ 0 & 0 & T_2 & 0 & \cdots \\ 0 & 0 & 0 & T_3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ I & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

are positive operators on $\mathcal{H}$,

$$AB - BA = \begin{bmatrix} T_1 & 0 & 0 & 0 & \cdots \\ 0 & T_2 - T_1 & 0 & 0 & \cdots \\ 0 & 0 & T_3 - T_2 & 0 & \cdots \\ 0 & 0 & 0 & T_4 - T_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with

$$\sigma(AB - BA) = \overline{\bigcup_{i=1}^{\infty} \sigma(T_i - T_{i-1})} \qquad (T_0 := 0).$$

Theorem (Bračič, Drnovšek, Farforovskaya, Rabkin, Zemánek 2009)

Let A and B be positive compact operators with a positive commutator $AB - BA$. Then $AB - BA$ is quasinilpotent and is contained in the radical of the Banach algebra generated by A and B.

Matrix case: $AB - BA$ is nilpotent and permutation similar to a strictly upper-triangular matrix.

Question

Is it enough to assume that one of A and B is compact?

Theorem

The positive commutator $AB - BA$ of A and B is quasinilpotent in the following cases:

- *Operators A and B are positive with one of them compact. (Gao 2014)*
- *$AB \geq BA \geq 0$ and AB (or BA) is power compact. (Drnovšek 2012)*

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Theorem (K., Šivic 2017a)

Let A and B be positive operators with a positive commutator. If one of them is compact, then $AB - BA$ is contained in the radical of the Banach algebra generated by them.

Question

Let A and B be compact operators on a Banach lattice X such that $AB \geq BA \geq 0$. Is the commutator $AB - BA$ contained in the radical of the Banach algebra generated by A and B?

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Theorem (K., Šivic 2017a)

- *If* $\dim X = 2$, *yes.*
- *On every Banach lattice* X *with* $\dim X \geq 3$ *there exist finite rank operators* A *and* B *such that* $AB \geq BA \geq 0$ *whereas their commutator* $AB - BA$ *is not contained in the radical.*

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$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad AB - BA = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $A(AB - BA)(e_1 + e_3) = e_1 + e_3$.

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Special cases

- Finite-dimensional case: commutator is nilpotent;
- Infinite-dimensional case: A or B is strictly singular $\Rightarrow AB - BA$ is strictly singular.

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Special cases

- Finite-dimensional case: commutator is nilpotent;
- Infinite-dimensional case: A or B is strictly singular $\Rightarrow AB - BA$ is strictly singular.

Theorem (Drnovšek, K. 2011)

If $AB - BA$ is invertible, then $0 \notin \rho_\infty(AB - BA)$.

Theorem (Shoda 1937, Albert and Muckenhoupt 1957)

A matrix C over a field is a commutator if and only if its trace is zero.

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Which positive matrices are positive commutators of positive matrices?

- If $A, B \geq 0$ and $AB - BA \geq 0$, then $AB - BA$ is nilpotent.

Theorem (Drnovšek, K. 2019)

Let C be an order continuous positive operator on a Banach lattice with the projection property.

- *The operator C is nilpotent if and only if there exists a positive central operator A and a positive order continuous nilpotent operator B such that $C = AB - BA$.*
- *If C is compact, then B can be chosen to be a compact operator.*

Theorem (Schneeberger, 1971)

Every compact operator on an infinite-dimensional separable Banach space $L^p(\mu)$ $(1 \leq p < \infty)$ *is a commutator.*

Question

Let C be a positive compact (quasinilpotent) operator na ℓ^p. Is C a commutator of positive operators?

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Theorem (Drnovšek, K. 2019)

Let $C = (c_{i,j})_{i,j=1}^{\infty}$ be a positive operator on ℓ^p $(1 \le p \le \infty)$ such that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sqrt{c_{ij}} < \infty$. Then the following are equivalent:

- *$C = AB - BA$ for some positive diagonal operator A na ℓ^p and some positive compact quasinilpotent operator B on ℓ^p;*
- *$C = AB - BA$ for some positive operators A and B on ℓ^p with one of them compact;*
- *C is quasinilpotent.*

Example

Let $(w_i)_{i=1}^{\infty}$ be a decreasing sequence of positive real numbers which converges to 0 slowly enough that the series $\sum_{i=1}^{\infty} w_i$ diverges. Then

$$C = \begin{bmatrix} 0 & w_1 & 0 & 0 & \cdots \\ 0 & 0 & w_2 & 0 & \cdots \\ 0 & 0 & 0 & w_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

defines a weighted shift on ℓ^2 which is a positive compact quasinilpotent operator.

- Operator C cannot be written as a commutator of positive operators with one of them diagonal.
- Operator C cannot be written as a commutator of positive operator with at least one of them compact and at least one of them quasinilpotent.
- Is C a commutator of positive operators with one of them compact?

Theorem (Drnovšek, K. 2019)

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Theorem

If $1 \leq p < \infty$, then every separable Banach lattice L^p is order and isometric isomorphic to one of the following Banach lattices:

- ℓ_n^p *or* ℓ^p;
- $L^p[0,1]$;
- $\ell_n^p \oplus L^p[0,1]$ *or* $\ell^p \oplus L^p[0,1]$.

Theorem (Drnovšek, K. 2019)

Suppose that the Banach lattice $X = (\oplus_{n=1}^{\infty} X_n)_p$ is an order Pelczyński decomposition of X, where $1 \leq p \leq \infty$. Let Y be a Banach lattice with the property that there exist positive operators $S : Y \to X$ and $T : X \to Y$ such that $\|S\| = 1$ and TS is the identity operator on Y. If C is a positive order continuous operator on $(Y \oplus X)_p$, then C is a commutator between two positive operators in each of the following cases:

1. *C is semi-compact, and X and X^* have order continuous norms.*
2. *C is semi-compact, C^* is AM-compact and X has order continuous norm.*
3. *C is AM-compact, C^* is semi-compact and X^* has order continuous norm.*
4. *C is compact, and either X or X^* have order continuous norms.*

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Finite-dimensional case

If $\dim X < \infty$, commutators have zero trace. Therefore, at least one eigenvalue λ of a commutator C is outside the disk $\{z \in \mathbb{C} : |z - 1| < 1\}$ yielding

$$\|C - I\| \geq r(C - I) \geq |\lambda - 1| \geq 1.$$

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Theorem (Popa 1982)

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that

$$\|[A, B] - I\| \leq \varepsilon$$

for some $\varepsilon > 0$. Then

$$\|A\| \cdot \|B\| \geq \tfrac{1}{2} \ln \tfrac{1}{\varepsilon}.$$

Popa's order analog (Drnovšek, K. 2025)

Let a and b be elements of a unital ordered normed algebra $\mathcal{A}$ with unit e. Suppose that at least one of the elements a and b is positive, and that for some $\varepsilon > 0$ there exists an element $x \in \mathcal{A}$ with $\|x\| \leq \varepsilon$ such that

$$[a, b] \geq e + x.$$

If the cone $\mathcal{A}^+$ is normal with normality constant α, then

$$\|a\| \cdot \|b\| \geq \tfrac{1}{2\alpha} \ln \tfrac{1}{\alpha\varepsilon}.$$

In particular, if the norm on $\mathcal{A}$ is monotone, then

$$\|a\| \cdot \|b\| \geq \tfrac{1}{2} \ln \tfrac{1}{\varepsilon}.$$

Theorem (Tao 2019)

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Then, for any $\varepsilon \in (0, 1/2)$, there exist operators $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$\|[A, B] - I\| \leq \varepsilon \qquad \textit{and} \qquad \|A\| \cdot \|B\| = O(\ln^5 \tfrac{1}{\varepsilon}).$$

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Tao's order analog (Drnovšek, K. 2025)

There exist positive operators $A, B \colon \ell^2 \to \ell^2$ such that $[A, B] = I + N$, where N is a nilpotent operator of nil-index 3. Furthermore, if $\varepsilon \in (0, 1)$, then A and B can be chosen in such a way that $\|A\| = O(\varepsilon^{-3})$, $\|B\| = O(\varepsilon^{-3})$ and $\|N\| = O(\varepsilon)$.

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Weiland-type result (Drnovšek, K. 2025)

Let a and b be elements of a unital ordered normed algebra $\mathcal{A}$ with a normal algebra cone $\mathcal{A}^+$. If $[a, b] \geq e$, then neither a nor b is either positive or negative.

- Let $\mathcal{H}$ be a Hilbert lattice and A a bounded operator on $\mathcal{H}$.
- An operator of the form $A^*A - AA^*$ is a self-commutator.
- If A is positive, then A^* is positive (in the sense of Banach lattices).
- Radjavi (1966): a self-adjoint operator A on a separable Hilbert space is a self-commutator if and only if zero lies within the convex hull of the essential spectrum of A.

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Question

What can be said about a positive self-commutator $A^*A - AA^* \geq 0$ of a positive operator A?

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Theorem (Drnovšek, K. 202⋆)

Let A be a positive operator on a Hilbert lattice $\mathcal{H}$ such that the self-commutator $C := A^*A - AA^*$ is also positive.

- *If A is idempotent, then $A^* = A$ and so $A^*A = AA^*$.*
- *If A is power-compact, then $A^*A = AA^*$.*

Theorem (Drnovšek, K. 202⋆)

Every positive compact central operator on a separable infinite-dimensional Hilbert lattice is a self-commutator of a positive operator.

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Why central operators?

- For every normal operator N on a Hilbert space, there exists a measure space (Ω, Σ, μ) and a function $\varphi \in L^\infty(\mu)$ such that N is unitarily equivalent to the multiplication operator M_φ acting on $L^2(\mu)$.
- If μ is σ-finite, then multiplication operators of the form M_φ, where $\varphi \in L^\infty(\mu)$ are precisely central operators on the Banach lattice $L^p(\mu)$ for $1 \leq p < \infty$.

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Theorem (Drnovšek, K. 202$\star$)

A band projection on ℓ^2 or $L^2[0,1]$ is a self-commutator of a positive operator if and only if its kernel is infinite-dimensional.

Mixed Case

The operator C on a Hilbert lattice $\mathcal{H} = \ell^2 \oplus L^2[0,1]$ represented with the 2×2 operator matrix

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

is a self-commutator of a positive operator on $\mathcal{H}$.

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There exists a positive isometry $X\colon \ell^2 \to L^2[0,1] \cong \bigoplus_{n=1}^{\infty} L^2[0,1]$ and a positive self-adjoint operator $Y\colon L^2[0,1] \to L^2[0,1]$ such that $Y^2 = XX^*$.

We define the positive operator Z as the infinite block operator matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ X & 0 & 0 & 0 & \dots \\ 0 & Y & 0 & 0 & \dots \\ 0 & 0 & Y & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We have $[Z^*, Z] = C$.

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There are no positive isometries from $L^2[0,1]$ to ℓ^2! (any such positive isometry should be a lattice homomorphism and so ℓ^2 should contain a closed sublatice isomorphic to $L^2[0,1]$ which is impossible as they are all closed linear spans of pairwise disjoint vectors: Bilokopytov and Troitsky 2023).

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Theorem (Drnovšek, K. 202$\star$)

Every positive central operator C on an infinite-dimensional separable Hilbert lattice is a sum of two positive self-commutators of positive operators.

Problem

Determine the dimension of the algebra generated by $n \times n$ matrices A and B?

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- The dimension of a commutative algebra of $n \times n$ matrices is at most $\lfloor \frac{n^2}{4} \rfloor + 1$. (Schur 1905)
- The dimension of the unital algebra generated by commuting $n \times n$ matrices A and B is of dimension at most n. (Gerstenhaber 1962)

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- The dimension of the unital algebra generated by commuting $n \times n$ matrices A and B is of dimension at most n. (Gerstenhaber 1962)

Problem

Let A and B be positive matrices with a positive commutator $AB - BA$. Determine the upper bound for the dimension of the unital algebra generated by A and B?

Theorem (K., Šivic 2017b)

- *The upper bound is* $\frac{n(n+1)}{2}$.
- *If A or B is irreducible, then the upper bound is n.*

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Special cases

- Let B be the diagonal matrix with strictly increasing positive diagonal entries and let $A = J_n$ be the $n \times n$ Jordan block. Then the upper bound $\frac{n(n+1)}{2}$ is attained;
- All dimensions up to n are attainable $(A = J_k, B = I)$.
- $B = C_n$ is a cycle of order $n \Rightarrow$ the algebra is n-dimensional;
- $B = P$ is a permutation matrix $\Rightarrow AP = PA$ and the dimension is at most n;
- All dimensions between n and $\frac{n(n+1)}{2}$ are attainable (Kolegov 2021)

Theorem (K., Šivic 2017b)

Let E and F be positive idempotent operators on a vector lattice X with a positive commutator $EF - FE$. Then the upper bound for the dimension of the unital algebra generated by E and F is

- 4 *if (E and E^*) or (F and F^*) are strictly positive and X is a Banach lattice.*
- 6 *if E or F is strictly positive or (X is a Banach lattice and E^* or F^* is strictly positive);*
- **9** *if X has the projection property and F is order continuous.*

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- If the semigroup generated by E and F consists of idempotent matrices, then the upper bound for the dimension is 7. (Drnovšek 2018)
- If E in F are positive idempotent operators on a Banach lattice with a positive commutator, then upper bound for the dimension of the unital algebra generated by E and F is 9.

Example

Consider the idempotents

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $EF - FE$ is positive and

$$\{I, E, F, EF, [E,F], [E,F]E, [E,F]F, [E,F]EF, [E,F]^2\}$$

and is a linearly independent set of generators for the unital algebra generated by E af F.