$\mathbb{L}$-functional analysis

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$\mathbb{L}$-functional analysis

$\mathbb{L}$-functional analysis is functional analysis but with $\mathbb{R}$ (or $\mathbb{C}$) replaced by a real (or complex) Dedekind complete unital f-algebra $\mathbb{L}$.

- Why is this useful/interesting?
- Why replace the scalars with a Dedekind complete unital f-algebra?

Probability theory in vector lattices

- Invented in 2000's in South Africa
- E (replacing L^1) is a Dedekind complete vector lattice, e (replacing $\mathbf{1}$) weak unit
- $T: E \to E$ **conditional expectation**: linear, positive, order continuous, $Te = e$, $R(T)$ Dedekind complete
- Extend T: $R(T)$ becomes universally complete
- $R(T)$ admits a very nice f-algebra multiplication
- E becomes an $R(T)$-module
- Define $\|\cdot\|_p : E \to R(T)$ by $\|f\|_p := T(|f|^p)^{\frac{1}{p}}$
- Define $L^p(T)$ as those f for which $\|f\|_p$ exists
- Azouzi, Kalauch, Kuo, Watson 2023: Completeness of $L^p(T)$, Riesz Representation Theorem for $L^2(T)$

Kaplansky-Hilbert modules

- An abelian AW*-algebra is a special abelian C*-algebra: Dedekind complete $C_{\mathbb{C}}(K)$ (i.e., K Stonean)

Kaplansky, 1953: initiated study of Kaplansky-Hilbert modules (KH-modules): Hilbert spaces H with $\mathbb{C}$ replaced by an abelian AW*-algebra $A \cong C_{\mathbb{C}}(K)$.

- H is an A-module
- $\langle \cdot, \cdot \rangle : H \times H \to A$, positive definite, A-sesquilinear
- Some completeness assumption

Kaplansky used KH-modules to characterize type I AW*-algebras.

- Edeko, Haasse, Kreidler (2024): A Decomposition Theorem for Unitary Group Representations on Kaplansky-Hilbert Modules and the Furstenberg-Zimmer Structure Theorem

Connection between those two theories?

- In probability, scalars: $R(T)$, universally complete VL
- In KH-modules, scalars: $A \cong C_{\mathbb{C}}(K)$, abelian AW*-algebra

Connection between those two theories?

- In probability, scalars: $R(T)$, universally complete VL
- In KH-modules, scalars: $A \cong C_{\mathbb{C}}(K)$, abelian AW*-algebra

Both are (real/complex) Dedekind complete unital f-algebras!

Our goal: unify both theories by setting up a general theory of functional analysis, replacing $\mathbb{R}$ (or $\mathbb{C}$) by a real (or complex) Dedekind complete unital f-algebra $\mathbb{L}$.

By representation theory we obtain

$$C(K) \subseteq \mathbb{L} \subseteq C^{\infty}(K)$$

$$C_{\mathbb{C}}(K) \subseteq \mathbb{L}_{\mathbb{C}} \subseteq C^{\infty}_{\mathbb{C}}(K)$$

In this talk we assume $\mathbb{L}$ is real (but our theory also covers the complex case).

Comparing $\mathbb{L}$ with $\mathbb{R}$

- From now on $\mathbb{L}$ is a fixed Dedekind complete unital f-algebra. Notation: $\lambda, \mu, 1 \in \mathbb{L}$.
- $\mathbb{L}$ plays the role of an **partially ordered ring** replacing $\mathbb{R}$

$\mathbb{R}$	$\mathbb{L}$
Field $0 \neq r$ is invertible Totally ordered	Commutative ring $0 \neq \lambda$ often not invertible Partially (lattice) ordered
metric/Dedekind complete	Dedekind complete

- real vector space $=$ $\mathbb{R}$-module: replaced by $\mathbb{L}$-module

Example

A Dedekind complete vector lattice E is an Orth(E)-module

- normed space: replaced by $\mathbb{L}$-normed space

$\mathbb{L}$-normed spaces

Definition

An $\mathbb{L}$-**normed space** $(X, \|\cdot\|)$ is an $\mathbb{L}$-module X equipped with a map $\|\cdot\| : X \to \mathbb{L}^+$ satisfying ($\lambda \in \mathbb{L}$, $x, y \in X$)

- $\|\lambda x\| = |\lambda| \, \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|x\| = 0 \Leftrightarrow x = 0$.

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An $\mathbb{L}$-normed space is an example of a **lattice normed space**, which goes back to Kantorovich (1936), who investigated mostly the non-module case.

Example

$(\mathbb{L}, |\cdot|)$ is an $\mathbb{L}$-normed space

Convergence

We define $x_\alpha \to x$ to mean that $\|x_\alpha - x\| \to 0$ in $\mathbb{L}$, so we need a notion of convergence in $\mathbb{L}$. **Order convergence** is used in both motivating examples.

Notation: $A \searrow 0$ means that $A \subseteq \mathbb{L}$ with $\inf A = 0$.

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Definition

Let X be an $\mathbb{L}$-normed space, (x_α) a net in X, and $x \in X$. Then we define $x_\alpha \to x$ to mean that

$$\exists \mathcal{E} \searrow 0 \; \forall \varepsilon \in \mathcal{E} \; \exists \alpha_0 \; \forall \alpha \geq \alpha_0 \;\; \|x_\alpha - x\| \leq \varepsilon.$$

Similar to convergence in $\mathbb{R}$, except $\mathcal{E}$ depends on the net (x_α). Note that notion of convergence in X is **not** topological! It turns X into a **convergence space**.

Completeness

Definition

A net (x_α) in an L-normed space X is **Cauchy** if $x_\alpha - x_\beta \to 0$. X is **complete** or an $\mathbb{L}$-**Banach space** if every Cauchy net converges.

The Dedekind completeness of $\mathbb{L}$ is equivalent to the completeness of $(\mathbb{L}, |\cdot|)$.

Thus the Dedekind completeness assumption on $\mathbb{L}$ is necessary.

Let S be a nonempty set.

Example

$$\ell_\infty(S, \mathbb{L}) := \{f\colon S \to \mathbb{L} \colon \exists M \in \mathbb{L}^+ \ \forall s \in S \ |f(s)| \leq M\}$$

Defining $(\lambda f)(s) := \lambda f(s)$ turns $\ell_\infty(S, \mathbb{L})$ into an $\mathbb{L}$-module, and for $f \in \ell_\infty(S, \mathbb{L})$, define (using Dedekind completeness of $\mathbb{L}$)

$$\|f\|_\infty := \sup_{s \in S} |f(s)|.$$

$\ell_\infty(S, \mathbb{L})$

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Theorem

$\ell_\infty(S, \mathbb{L})$ is an $\mathbb{L}$-Banach space.

Proof is **very** similar to the classical case.

- We prove similar results for $\mathbb{L}$-valued ℓ^p and c_0.

Operators

- X, Y $\mathbb{L}$-normed spaces, $T \in \mathrm{Hom}_{\mathbb{L}}(X, Y)$.

Definition

T is **bounded** if $\exists M \in \mathbb{L}^+ \ \forall x \in X \ \ \|Tx\|_Y \leq M \|x\|_X$.

$$\|T\| := \inf\{M \in \mathbb{L}^+ \colon \forall x \in X \ \ \|Tx\| \leq M \|x\|\}$$

$$B(X, Y) := \{T \in \mathrm{Hom}_{\mathbb{L}}(X, Y) \colon \ T \text{ is bounded}\}.$$

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Theorem

$B(X, Y)$ is an $\mathbb{L}$-normed space satisfying $\|TS\| \leq \|T\| \|S\|$ which is complete whenever Y is complete.

Proof is very similar to the classical case.

Idempotents and disjointness

Let $\mathbb{P}$ be the Boolean algebra of idempotents in $\mathbb{L}$:

Definition

$$\mathbb{P} := \{\pi \in \mathbb{L} \colon \pi^2 = \pi\}$$

For $\pi \in \mathbb{P}$, $\pi^c := 1 - \pi \in \mathbb{P}$.

- $\mathbb{P}$ consists of components (fragments) of $1 \in \mathbb{L}$
- The band projections in $\mathbb{L}$ are $\lambda \mapsto \pi\lambda$ for $\pi \in \mathbb{P}$
- In $\mathbb{L}$: $\lambda \perp \mu \Leftrightarrow \exists \pi \in \mathbb{P} \colon \lambda = \pi\lambda$ and $\mu = \pi^c\mu$

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This disjointness structure can be transferred to $\mathbb{L}$-modules, even though $\mathbb{L}$-modules (like vector spaces) need not be ordered:

Definition

Let X be an $\mathbb{L}$-module. We define $x, y \in X$ to be ~~disjoint~~ **separated** if there is $\pi \in \mathbb{P}$ with $x = \pi x$ and $y = \pi^c y$.

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The $\mathbb{L}$-module $X = \mathbb{L}^2$ has two additional structures:

- $\mathbb{L}$-vector lattice: $(\lambda_1, \lambda_2) \vee (\mu_1, \mu_2) = (\lambda_1 \vee \mu_1, \lambda_2 \vee \mu_2)$
- $\mathbb{L}$-inner product space: $\langle (\lambda_1, \lambda_2), (\mu_1, \mu_2) \rangle = (\lambda_1 \mu_1, \lambda_2 \mu_2)$

Let $x, y \in X$.

- x, y separated $\Rightarrow |x| \wedge |y| = 0 \Rightarrow \langle x, y \rangle = 0$
- $\langle (1,1), (1,-1) \rangle = 0$ but $|(1,1)| \wedge |(1,-1)| \neq 0$
- $(1,0) \wedge (0,1) = 0$ but they are not separated

So for $\mathbb{L}$-vector lattices, separatedness is different from disjointness (and they are also different from orthogonality for $\mathbb{L}$-inner product modules).

Is this notion of separatedness useful?

Let X and Y be $\mathbb{L}$-modules. A map $\varphi\colon X \to Y$ is $\mathbb{P}$-**homogeneous** if $\varphi(\pi x) = \pi\varphi(x)$ for all $\pi \in \mathbb{P}$.

Lemma

If $\varphi\colon X \to Y$ is $\mathbb{P}$-homogeneous and x and y are separated, then

$$\varphi(x + y) = \varphi(x) + \varphi(y).$$

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Proof.

Pick $\pi \in \mathbb{P}$ with $\pi x = x$ and $\pi^c y = y$, then $\pi^c x = 0$ and $\pi y = 0$

$$\begin{aligned}\varphi(x+y) &= (\pi + \pi^c)\varphi(x+y) = \pi\varphi(x+y) + \pi^c\varphi(x+y)\\ &= \varphi(\pi x + \pi y) + \varphi(\pi^c x + \pi^c y)\\ &= \varphi(\pi x) + \varphi(\pi^c y)\\ &= \varphi(x) + \varphi(y)\end{aligned}$$

□

So $\mathbb{P}$-homogeneous maps are additive on separated elements. Is that useful?

The norm is $\mathbb{P}$-homogeneous, so for separated x and y in an $\mathbb{L}$-normed space:

$$\|x + y\| = \|x\| + \|y\|$$

So an $\mathbb{L}$-normed space is somewhat similar to an AL-space.

Thus, if $\pi \in \mathbb{P}$, then

$$X = \pi X \oplus_1 \pi^c X.$$

This is cute but maybe not that useful. However...

Hahn-Banach

$\varphi\colon X \to \mathbb{L}$ is **sublinear** if $\varphi(\lambda x) = \lambda\varphi(x)$ and $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ for $\lambda \in \mathbb{L}^+$ and $x, y \in X$.

Theorem

Let X be a real $\mathbb{L}$-module, $Y \subseteq X$ submodule, $f \in \mathrm{Hom}_{\mathbb{L}}(Y, \mathbb{L})$, $\varphi\colon X \to \mathbb{L}$ sublinear with $f(y) \leq \varphi(y)$ for all $y \in Y$. Then there exists an $F \in \mathrm{hom}_{\mathbb{L}}(X, \mathbb{L})$ extending f with $F(x) \leq \varphi(x)$ for all $x \in X$.

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Let X be a real $\mathbb{L}$-module, $Y \subseteq X$ submodule, $f \in \mathrm{Hom}_{\mathbb{L}}(Y, \mathbb{L})$, $\varphi \colon X \to \mathbb{L}$ sublinear with $f(y) \leq \varphi(y)$ for all $y \in Y$. Then there exists an $F \in \mathrm{hom}_{\mathbb{L}}(X, \mathbb{L})$ extending f with $F(x) \leq \varphi(x)$ for all $x \in X$.

Classical proof relies on the fact that if $\lambda \neq 0$, then (λ is invertible) and ($\lambda > 0$ or $\lambda < 0$). Neither hold in $\mathbb{L}$.

- One can approximating $\lambda \neq 0$ by invertibles
- One can write $\lambda x = \lambda^+ x - \lambda^- x$ and a crucial step of the proof is to use the additivity of φ on the separated $\lambda^+ x$ and $\lambda^- x$

Is $\mathbb{L}$-Hahn-Banach useful?

Corollary

Let X be an $\mathbb{L}$-normed space. Then $X^* := B(X, \mathbb{L})$ separates the points of X, and $J\colon X \to X^{**}$ is isometric.

Corollary

The completion of X can be defined as $\overline{J(X)}$ in X^{**}.

This circumvents set-theoretic issues with having to consider equivalence classes of Cauchy nets.

Let X be an $\mathbb{L}$-normed space.

Definition

For $x \in X$, the **support** $\pi_x \in \mathbb{P}$ of x is defined by

$$\pi_x := \inf\{\pi \in \mathbb{P} \colon \pi x = x\} = \min\{\pi \in \mathbb{P} \colon \pi x = x\}.$$

- The last equality need not hold in non-normed $\mathbb{L}$-modules
- x and y are separated $\Leftrightarrow \pi_x \wedge \pi_y = 0$

Definition

For an $\mathbb{L}$-normed space X, define the support of X

$$\pi_X := \sup_{x \in X} \pi_x \in \mathbb{P}.$$

π_X is the smallest idempotent acting as 1 on X.
In general there is no $x \in X$ with $\|x\| = \pi_X$, but...

Theorem

Let X be an $\mathbb{L}$-Banach space. Then there is an $x \in X$ with $\|x\| = \pi_X$

Proof is the same as the proof for KH-modules from Edeko, Haasse, Kreidler (2024).

Proof.

Define $x \preceq y$ to mean $x = \pi_x y$. If $X = \mathbb{L}$, then $\lambda \preceq \mu$ precisely when λ is a **fragment** (component) of μ. In 2020, Mykhaylyuk, Pliev, and Popov started studying $\preceq$ on vector lattices (therein called 'lateral order').

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Consider $S = \{x \in X : \|x\| \in \mathbb{P}\}$; we want to show that S has a $\preceq$-maximal element x. Let K be a chain in S. Indexed by itself, K turns out to be a Cauchy net, and its limit turns out to be an upper bound for K. By Zorn, S has a maximal element x which turns out to satisfy $\|x\| = \pi_X$. □

Is this useful?

Let H be an $\mathbb{L}$-module.

Definition

An **inner product** is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{L}$ satisfying

- $\langle x, x \rangle \in \mathbb{L}^{+}$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$

Note that $\|x\| := \sqrt{\langle x, x \rangle}$ turns H into an $\mathbb{L}$-normed space; if it is complete, H is called an $\mathbb{L}$-**Hilbert space**.

If $S \subseteq H$ then the orthogonal complement $S^{\perp} \subseteq H$ is closed and hence complete.

- Does H have an orthonormal basis?

Suborthonormal basis

Let H be an $\mathbb{L}$-Hilbert space. Note that if $\pi_H \neq 1$, there is no $x \in H$ with $\|x\| = 1$.

Definition

$S \subseteq H$ is a **suborthonormal basis** if $\langle x, y \rangle = 0$ and $\langle x, x \rangle \in \mathbb{P} \setminus \{0\}$ for all $x \neq y \in S$, and $S^{\perp} = \{0\}$

Theorem

Let H be an $\mathbb{L}$-Hilbert space. Then there is an ordinal γ and a suborthonormal basis $(b_\alpha)_{\alpha\in\gamma}$ of H such that $\alpha \mapsto \|b_\alpha\|$ is decreasing.

Recall that for each $\mathbb{L}$-Banach X space there is an $x \in X$ with maximal idempotent norm ($\|x\| = \pi_X$); in particular this holds for closed subspaces of an $\mathbb{L}$-Hilbert space.

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Proof.

Pick a b_0 with $\|b_0\| = \pi_H$. For $0 \neq \beta \in \gamma$, by Transfinite Recursion pick b_β in $H_\beta := \{b_\alpha : \alpha < \beta\}^\perp$ with $\|b_\beta\| = \pi_{H_\beta}$ (if $H_\beta \neq \{0\}$). Since H_β is decreasing, $\beta \mapsto \|b_\beta\|$ is decreasing. By a cardinality argument this process must stop at some ordinal γ when $\{b_\alpha : \alpha < \gamma\}^\perp = \{0\}$. □

This is used to prove an ℓ^2-representation theorem for $\mathbb{L}$-Hilbert spaces.

Every $\mathbb{L}$-Hilbert space is a direct sum of ℓ^2-spaces taking values into disjoint parts of $\mathbb{L}$.

Theorem

Let H be an $\mathbb{L}$-Hilbert space. Then there is a disjoint collection of idempotents $(\pi_i)_{i\in I}$ in $\mathbb{L}$ and sets $(S_i)_{i\in I}$ such that

$$H \cong \bigoplus_{i\in I} \ell^2(S_i, \pi_i \mathbb{L})$$

The $\pi_i's$ correspond to the jumps in $\alpha \mapsto \|b_\alpha\|$ from the previous theorem.

Conclusion and prospects

$\mathbb{L}$-functional analysis is a nice theory, simultaneously generalizing the basic setup of KH-modules and parts of probability in vector lattices.

Some related work:

- Jiang, van der Walt, W.: $\mathbb{L}$-valued integration
- Zhang, Yan, Liu: $\mathbb{L}$-Bochner spaces w.r.t. scalar-values measures (arxiv)
- Chamberlain: $\mathbb{L}$-vector lattices

Future work:

- Generalize the rest of functional analysis to $\mathbb{L}$-functional analysis;
- Find a nice application.

Thank you for your attention!