A construction of a universal completion

(Axiom of choice-free construction)

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06 JUIN 2025

Positivity XII

Plan

1. Introduction :
2. From non empty set to Riesz space : Veksler's construnction
3. Truncated Riesz space
4. The construction of the universal completion
5. The universal completion

Definitions

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- The construction follows the general method described by Veksler in his paper :A.I. Veksler, Embedding of lattice cones in Riesz spaces,

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The quotient space

Definition

Let C be a cone and define the relation $\sim$ on $C \times C$ by

$$(x_1, x_2) \sim (x_3, x_4) \iff x_1 + x_4 + x = x_2 + x_3 + x, \text{ for some } x \in C.$$

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As usual, we introduce the quotient space

$$E^C = C \times C \big/ \sim .$$

To simplify notation, an elemnt of E^C will be denoted by $[x, y]$.

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$$(M\ 11) : u \leq v \text{ in } C \text{ implies that } v = u + w \text{ for some } w \in C,$$

then E^C is a Riesz space, and the positive cone $(E_C)^+$ of E^C is the lattice cone C.

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Notice that on a Riesz space we can define many truncations, for example, by considering $\tau(x) = x \wedge a$, for any $a \in E^{+}$. Hence, to differentiate between truncations, we introduce the following notion of fixed points of a truncation.

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Theorem

Two truncations τ_1 and τ_2 on a Riesz space E coincide if and only if $Fix\left(\tau_1\right) = Fix\left(\tau_2\right).$

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It's not difficult to see that, if E is Archimedean, for any $0 < a \in E$, the truncation defined by $x \mapsto x \wedge a$ is in $T_* E$, since E is assumed to be archimedean.

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1. *Any Archimedean truncation on a universally complete vector lattice is unital*
2. *Assume that E is Archimedean and let e be a distinguished weak unit in E^u. A unary operation τ on E^+ is a truncation if and only if there exists a component p of e in E^u and $u \in E^u$ such that $p \wedge u = 0$ and $\tau(x) = px + u \wedge x$ for all $x \in E^+$.*

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3. *If E is Dedekind complete, then E is universally complete if and only if, any archimedean truncation on E is unital.*

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Archimedean truncation VS Archimedean vector lattice

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2 From non empty set to Riesz space : Veksler's construnction

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5 The universal completion

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Theorem

The set T_*E equipped with the binairy operation $+ : T_*E \times T_*E \longrightarrow T_*E : (\tau_1, \tau_2) \longmapsto \tau_1 + \tau_2$ is a commutative monoid with the neutral element for addition

$$e := 0 : E^+ \longrightarrow E^+ : x \mapsto 0.$$

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The commutative monoid $(T_*E, +)$ *equipped with the map :*

$$. : \mathbb{R}^+ \times T_*E \longrightarrow T_*E : (\alpha, \tau) \longmapsto \alpha.\tau$$

is a cone.

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Thank you for your attention