# A construction of a universal completion

(Axiom of choice-free construction)

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## Plan

- Introduction :
- 2 From non empty set to Riesz space : Veksler's construnction
- Truncated Riesz space
- 4 The construction of the universal completion
- 5 The universal completion

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(Vulikh) Assume that two universally complete Riesz spaces L and M include two order dense Riesz subspaces  $L_1$  and  $M_1$ , respectively, that are Riesz isomorphic, say via  $\pi: L_1 \longrightarrow M_1$ . Then,  $\pi$  extends uniquely to a Riesz isomorphism from L to M.

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- The construction follows the general method described by Veksler in his paper :A.I. Veksler, Embedding of lattice cones in Riesz spaces,

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The neutral element e will be denoted by 0.

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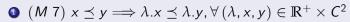
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## The quotient space

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Let C be a cone and define the relation  $\sim$  on  $C \times C$  by

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As usual, we introduce the quotient space

$$E^C = C \times C / \sim$$
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To simplify notation, an elemnt of  $E^{C}$  will be denoted by [x, y].

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then  $E^{C}$  is a Riesz space, and the positive cone  $(E_{C})^{+}$  of  $E^{C}$  is the lattice cone C.

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#### Theorem

Two truncations  $\tau_1$  and  $\tau_2$  on a Riesz space E coincide if and only if  $Fix(\tau_1) = Fix(\tau_2)$ .

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It's not difficult to see that, if E is Archimedean, for any  $0 < a \in E$ , the truncation defined by  $x \mapsto x \wedge a$  is in  $T_*E$ , since E is assumed to be archimedean.

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### $\mathsf{Theorem}_{\mathsf{l}}$

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- Assume that E is Archimedean and let e be a distinguished weak unit in  $E^u$ . A unary operation  $\tau$  on  $E^+$  is a truncation if and only if there exists a component p of e in  $E^u$  and  $u \in E^u$  such that  $p \wedge u = 0$  and  $\tau(x) = px + u \wedge x$  for all  $x \in E^+$ .

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- (a) If E is Dedekind complete, then E is universally complete if and only if, any archimedean truncation on E is unital.

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② Let  $(E, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. The lattice-ordered algebra of all (equivalence classes of) measurable real-valued functions on E is denoted by  $L_0(\mu)$  is an Archimedean Riesz space,

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② Let  $(E, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. The lattice-ordered algebra of all (equivalence classes of) measurable real-valued functions on E is denoted by  $L_0(\mu)$  is an Archimedean Riesz space, while the truncation defined by  $\tau(x) = 1_A x + 1_{E \setminus A} x$  for some measurable set A in E is not always an Archimedean truncation.

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$$(\tau_1, \tau_2) \in T_*E \times T_*E : \forall x \in E^+, (\tau_1 + \tau_2)(x) = \bigvee_{y \in Fix(\tau_1) + Fix(\tau_2)} (x \wedge y).$$

In what follows, E will be assumed to be Dedekind complete. First, we will start by defining addition in  $T_*E$  in the following manner:

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#### **Theorem**

The set  $T_*E$  equipped with the binairy operation

 $+: T_*E \times T_*E \longrightarrow T_*E: (\tau_1, \tau_2) \longmapsto \tau_1 + \tau_2$  is a commutative monoid with the neutral element for addition

$$e := 0 : E^+ \longrightarrow E^+ : x \mapsto 0$$



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For  $au \in \mathcal{T}_* E$  and  $\lambda \in [0, +\infty)$  , we define  $\alpha. au$  by

$$\forall x \in E^+, (\alpha.\tau)(x) = \begin{cases} \alpha\tau(\alpha^{-1}x) & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$$

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#### Theorem

The commutative monoid  $(T_*E, +)$  equipped with the map :

$$: \mathbb{R}^+ \times T_*E \longrightarrow T_*E : (\alpha, \tau) \longmapsto \alpha.\tau$$

is a cone.



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• For  $y \in E^+$ , let  $A(y) = \{\tau(a) - \tau(y), a \ge y\}$  and  $\tau_y$  is the truncation defined by  $x \mapsto \bigvee_{Fix(\tau) - \tau(y)} x \land a$ .

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## Plan

- Introduction
- 2 From non empty set to Riesz space : Veksler's construnction
- Truncated Riesz space
- 4 The construction of the universal completion
- 5 The universal completion

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The map  $E^+ \longrightarrow (E^T)^+ : x \mapsto [\tau_x, 0]$  can be extended to one to one Riesz homomorphism with order dense range where  $\tau_x(a) = x \wedge a$ .

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# Thank you for your attention