Banach lattices of positively homogeneous functions induced by a Banach space

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Our "universe": positively homogeneous functions on a dual Banach space

Throughout, E will be a real Banach space, with (continuous) dual E^*.

We work within the vector lattice of positively homogeneous functions on E^*:

$$H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*) \text{ for } \lambda \in [0, \infty) \text{ and } x^* \in E^*\}.$$

Reason: We are interested in vector lattices generated by the evaluation maps

$$\delta_x : \; E^* \to \mathbb{R}, \quad x^* \mapsto \langle x, x^* \rangle \qquad (x \in E).$$

They are linear, so vector lattice combinations of them are positively homogeneous.

Simple examples of $H[E] = \{f\colon E^* \to \mathbb{R} : f \text{ is positively homogeneous}\}$

Recall: E is a real Banach space with dual E^*.

Definition.

$$S_{E^*} := \{x^* \in E^* : \|x^*\| = 1\} \quad \text{and} \quad \mathbb{R}^{S_{E^*}} := \{f\colon S_{E^*} \to \mathbb{R}\}.$$

Observation. The restriction mapping

$$R\colon \; H[E] \to \mathbb{R}^{S_{E^*}}, \qquad f \mapsto f\restriction_{S_{E^*}},$$

is a lattice isomorphism (provided $E \neq \{0\}$).

Low-dimensional examples.

- For $E = \{0\}$, we have $E^* = \{0\}$, so $H[\{0\}] = \{0\}$.
- For $E = \mathbb{R}$, we have $E^* = \mathbb{R}$, so $S_{E^*} = \{\pm 1\}$, and hence

$$\begin{aligned} H[\mathbb{R}] &= \{f\colon \mathbb{R} \to \mathbb{R} : f(\lambda) = \lambda f(1) \text{ and } f(-\lambda) = \lambda f(-1) \text{ for } \lambda \in [0,\infty)\} \\ &\cong \mathbb{R}^2. \end{aligned}$$

- For $E = \ell_2^2$, we have $E^* = \ell_2^2$, so $S_{E^*} = S^1\; (= \{(s,t) \in \mathbb{R}^2 : s^2 + t^2 = 1\})$; that is, $H[\ell_2^2] = \mathbb{R}^{S^1}$ is already huge!

Finding a smaller universe: the Banach lattice $H^p[E]$

Let $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$.

Definition. The **weak p-summing norm** of an n-tuple $(x_j^*)_{j=1}^n \in (E^*)^n$ is the operator norm of the operator

$$E \to \ell_p^n, \quad x \mapsto (\langle x, x_j^* \rangle)_{j=1}^n;$$

that is,

$$\|(x_j^*)_{j=1}^n\|_{p,\mathrm{weak}} := \sup_{\|x\| \leqslant 1} \Big(\sum_{j=1}^n |\langle x, x_j^* \rangle|^p \Big)^{\frac{1}{p}}.$$

Note: $\big(\sum_{j=1}^n |t_j|^p\big)^{\frac{1}{p}} = \max_{1 \leqslant j \leqslant n} |t_j|$ for $p = \infty$ by convention.

We use it to define, for $f \in H[E]$,

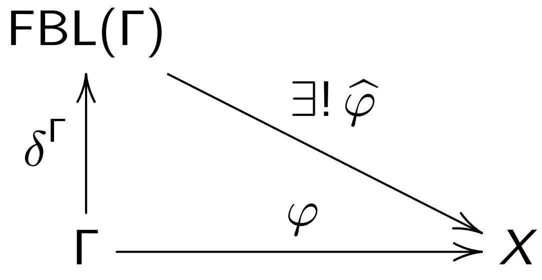
$$\|f\|_{\mathrm{FBL}^p[E]} := \sup \Big\{ \Big(\sum_{j=1}^n |f(x_j^*)|^p \Big)^{\frac{1}{p}} : \\ n \in \mathbb{N},\ (x_j^*)_{j=1}^n \in (E^*)^n,\ \|(x_j^*)_{j=1}^n\|_{p,\mathrm{weak}} \leqslant 1 \Big\} \in [0, \infty].$$

Prop. $H^p[E] := \{ f \in H[E] : \|f\|_{\mathrm{FBL}^p[E]} < \infty \}$ is a vector sublattice of $H[E]$, and a Banach lattice with respect to the norm $\|\cdot\|_{\mathrm{FBL}^p[E]}$.

Origin: the **free p-convex Banach lattice** generated by E.

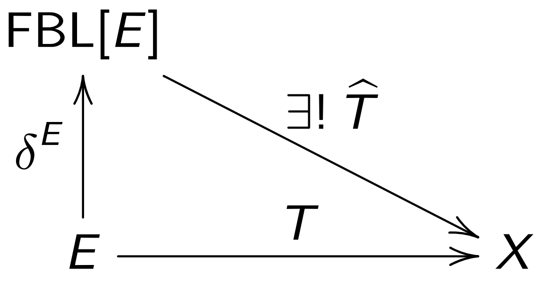
Free Banach lattices — a brief introduction

The free Banach lattice generated by a set Γ (De Pagter, Wickstead 2015): There is a Banach lattice $\mathrm{FBL}(\Gamma)$ and a bounded map $\delta^\Gamma : \Gamma \to \mathrm{FBL}(\Gamma)$ such that, for every Banach lattice X and every bounded map $\varphi\colon \Gamma \to X$, there is a unique lattice homomorphism $\widehat{\varphi}$ such that



and $\quad \|\widehat{\varphi}\| = \sup_{\gamma\in\Gamma}\|\varphi(\gamma)\|.$

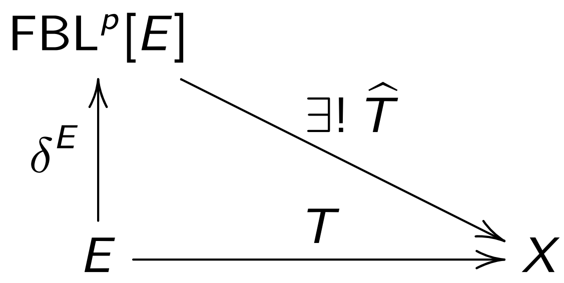
The free Banach lattice generated by a Banach space E (Avilés, Rodríguez, Tradacete 2018): There is a Banach lattice $\mathrm{FBL}[E]$ and a linear isometry $\delta^E\colon E \to \mathrm{FBL}[E]$ such that, for every Banach lattice X and every bounded linear map $T\colon E \to X$, there is a unique lattice homomorphism $\widehat{T}$ such that



and $\quad \|\widehat{T}\| = \|T\|.$

Correspondence: $\mathrm{FBL}(\Gamma) = \mathrm{FBL}[\ell_1(\Gamma)]$.

The free ***p*-convex Banach lattice generated by a Banach space** E, for $1 \leqslant p \leqslant \infty$ (Jardón-Sánchez, L, Taylor, Tradacete, Troitsky 2022): There is a p-convex Banach lattice $\mathrm{FBL}^p[E]$ and a linear isometry $\delta^E\colon E \to \mathrm{FBL}^p[E]$ such that, for every p-convex Banach lattice X and every bounded linear map $T\colon E \to X$, there is a unique lattice homomorphism $\widehat{T}$ such that



and $\|\widehat{T}\| \leqslant M\|T\|$,

where M denotes the p-convexity constant of X.

Correspondence: $\mathrm{FBL}[E] = \mathrm{FBL}^1[E]$.

Key result: $\mathrm{FBL}^p[E]$ is a function lattice; very useful in applications.

Construction of $\mathrm{FBL}^p[E]$ for $1 \leqslant p \leqslant \infty$

Recall: For $x \in E$, $\delta_x \colon E^* \to \mathbb{R}$ is the evaluation map:

$$\delta_x(x^*) = \langle x, x^* \rangle.$$

It is (positively) homogeneous and satisfies

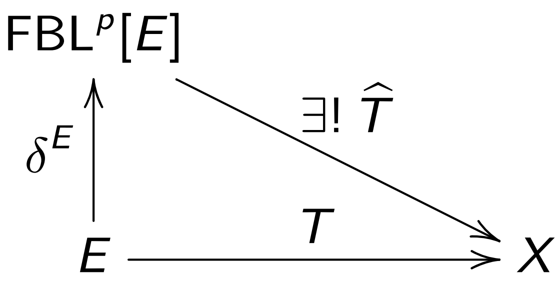
$$\|\delta_x\|_{\mathrm{FBL}^p[E]} = \|x\|,$$

so $\delta_x \in H^p[E]$.

Def'n. $\mathrm{FBL}^p[E]$ is the closed vector sublattice of $H^p[E]$ gen. by $\{\delta_x : x \in E\}$,

$$\delta^E \colon \; E \to \mathrm{FBL}^p[E], \quad x \mapsto \delta_x.$$

Theorem (J-S,L,T,T,T). $(\mathrm{FBL}^p[E], \delta^E)$ is the free p-convex Banach lattice generated by E; that is, $\mathrm{FBL}^p[E]$ is a p-convex Banach lattice and δ^E a linear isometry; and for every p-convex Banach lattice X and every bounded linear map $T \colon E \to X$, there is a unique lattice homomorphism $\widehat{T}$ such that



and $\qquad \|\widehat{T}\| \leqslant M\|T\|,$

where M denotes the p-convexity constant of X.

Finite-dimensional Banach spaces

Recall:

- $S_{E^*} := \{x^* \in E^* : \|x^*\| = 1\}$,
- the restriction mapping

$$R\colon\ H[E] \to \mathbb{R}^{S_{E^*}}, \quad f \mapsto f\restriction_{S_{E^*}},$$

  is a lattice isomorphism (provided $E \neq \{0\}$).

Prop. Let E be a non-zero, finite-dimensional Banach space and $1 \leqslant p \leqslant \infty$. Then

$$R(\mathrm{FBL}^p[E]) = C(S_{E^*}) \quad \text{and} \quad R(H^p[E]) = \ell_\infty(S_{E^*}).$$

Obs. $C(S_{E^*}) = \ell_\infty(S_{E^*}) \iff S_{E^*}$ is discrete $\iff \dim E \leqslant 1$.

Corollary. $\mathrm{FBL}^p[E] = H^p[E] \iff \dim E \leqslant 1$.

Aim: Explore the "gap" between $\mathrm{FBL}^p[E]$ and $H^p[E]$.

Recall:

- E is a real Banach space,
- $H[E] := \{\text{positively homogeneous functions } E^* \to \mathbb{R}\}$,
- $H^p[E] := \{f \in H[E] : \|f\|_{\mathrm{FBL}^p[E]} < \infty\}$ for $1 \leqslant p \leqslant \infty$,
- for $x \in E$, $\delta_x \colon E^* \to \mathbb{R}$ is the evaluation map: $\delta_x(x^*) = \langle x, x^* \rangle$,
- $\mathrm{FBL}^p[E]$ is the closed vector sublattice of $H^p[E]$ gen. by $\{\delta_x : x \in E\}$.

Definition. $I[E]$ is the order ideal of $H[E]$ generated by $\{\delta_x : x \in E\}$.

Obs.

$$I[E] = \left\{ f \in H[E] : |f| \leqslant \bigvee_{j=1}^{n} |\delta_{x_j}| \text{ for some } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in E \right\} \subseteq H^p[E].$$

Theorem (L, Tradacete). Let $1 \leqslant p < \infty$. The following are equivalent for a Banach space E:

- $\dim E < \infty$,
- $I[E] = H^p[E]$,
- $I[E]$ is closed in $H^p[E]$,
- $\mathrm{FBL}^p[E] \subseteq I[E]$.

An intermediate vector lattice & more characterizations of $\dim E < \infty$

Recall: $B_{E^*} := \{x^* \in E^* : \|x^*\| \leqslant 1\}$ is weak*-compact (Banach–Alaoglu).

Definition. $H_{w^*}[E] := \{f \in H[E] : f\restriction_{B_{E^*}}$ is weak*-continuous$\}$.

Obs.

- $H_{w^*}[E]$ is a sublattice of $H[E]$,
- $H_{w^*}[E] \cap H^p[E]$ is closed in $(H^p[E], \|\cdot\|_{\mathrm{FBL}^p[E]})$,
- $\mathrm{FBL}^p[E] \subseteq H_{w^*}[E]$.

Theorem (continued; L, Tradacete). Let $1 \leqslant p < \infty$. The following are equivalent for a Banach space E:

- $\dim E < \infty$,
- $I[E] = H^p[E]$,
- $I[E]$ is closed in $H^p[E]$,
- $\mathrm{FBL}^p[E] \subseteq I[E]$,
- $\mathrm{FBL}^p[E] = H_{w^*}[E]$,
- $H_{w^*}[E] \subseteq H^p[E]$,
- $H_{w^*}[E] \subseteq I[E]$,
- $I[E] \cap H_{w^*}[E]$ is closed in $H^p[E]$.

Corollary. For $1 \leqslant p < \infty$ and $2 \leqslant \dim E < \infty$,

$$C(S_{E^*}) \cong \mathrm{FBL}^p[E] = H_{w^*}[E] \subsetneq I[E] = H^p[E] \cong \ell_\infty(S_{E^*}).$$

Recall:

- $\mathrm{FBL}^p[E]$ is the closed vector sublattice of $H^p[E]$ gen. by $\{\delta_x : x \in E\}$,
- $I[E]$ is the ideal of $H[E]$ generated by $\{\delta_x : x \in E\}$, and
- $H_{w^*}[E] := \{f \in H[E] : f\restriction_{B_{E^*}}$ is weak*-continuous$\}$.

Definition. $I_{w^*}[E] := I[E] \cap H_{w^*}[E]$.

Recall: $\dim E < \infty \iff I_{w^*}[E] = H_{w^*}[E] \iff I_{w^*}[E]$ is closed in $H^p[E]$.

Hence, for $\dim E = \infty$, we consider its closure $\overline{I_{w^*}[E]}$ in $(H^p[E], \|\cdot\|_{\mathrm{FBL}^p[E]})$.

Question. When is $\mathrm{FBL}^p[E] = \overline{I_{w^*}[E]}$?

Conjecture. $\dim E < \infty \iff \mathrm{FBL}^p[E] = \overline{I_{w^*}[E]}$.

Recall: $\dim E < \infty \iff \mathrm{FBL}^p[E] = H_{w^*}[E]$.

This verifies "$\Longrightarrow$" of the conjecture.

An "almost-proof" of the conjecture

Recall:

- $I_{w^*}[E] := I[E] \cap H_{w^*}[E]$, where
- $I[E]$ is the ideal of $H[E]$ generated by $\{\delta_x : x \in E\}$, and
- $H_{w^*}[E] := \{f \in H[E] : f\restriction_{B_{E^*}}$ is weak*-continuous$\}$.
- $\dim E < \infty \implies \mathrm{FBL}^p[E] = \overline{I_{w^*}[E]}$.
- Question: is $\Longleftarrow$ true?

Theorem (L, Tradacete). Let $1 \leqslant p < \infty$ and E a Banach space which admits an ∞-dimensional, separable quotient space. Then

$$I_{w^*}[E] \not\subseteq \mathrm{FBL}^p[E],$$

so

$$\mathrm{FBL}^p[E] \subsetneq \overline{I_{w^*}[E]}.$$

(In)famous open question. Does every ∞-dimensional Banach space admit an ∞-dimensional, separable quotient space?

Lattice homomorphisms and positively homogeneous maps

Recall: The **weak p-summing norm** of $(x_j^*)_{j=1}^n \in (E^*)^n$ is

$$\|(x_j^*)_{j=1}^n\|_{p,\mathrm{weak}} := \sup_{\|x\|\leqslant 1}\left(\sum_{j=1}^n |\langle x, x_j^*\rangle|^p\right)^{\frac{1}{p}}.$$

Lemma (L, Tradacete). Let $1 \leqslant p < \infty$, E and F Banach spaces, and $\Phi\colon F^* \to E^*$ a positively homogeneous map.

- The composition operator

$$C_\Phi\colon f \mapsto f \circ \Phi$$

defines a lattice homomorphism $C_\Phi\colon H[E] \to H[F]$.

- Suppose that

$$\|\Phi\|_p := \sup\Big\{\big\|\big(\Phi(y_j^*)\big)_{j=1}^m\big\|_{p,\mathrm{weak}} : m \in \mathbb{N},\ (y_j^*)_{j=1}^m \in (F^*)^m,\ \|(y_j^*)_{j=1}^m\|_{p,\mathrm{weak}} \leqslant 1\Big\} < \infty.$$

Then $C_\Phi(H^p[E]) \subseteq H^p[F]$, and the restriction

$$C_\Phi\colon \ (H^p[E], \|\cdot\|_{\mathrm{FBL}^p[E]}) \to (H^p[F], \|\cdot\|_{\mathrm{FBL}^p[F]}), \quad f \mapsto f \circ \Phi,$$

is bounded with norm $\|\Phi\|_p$.

Lemma (continued)

Recall: Let $1 \leqslant p < \infty$, E and F Banach spaces, and $\Phi\colon F^* \to E^*$ a positively homogeneous map.

- $C_\Phi\colon H[E] \to H[F]$, $f \mapsto f \circ \Phi$, is a lattice homomorphism.
- Suppose that

$$\|\Phi\|_p := \sup\Big\{ \big\| \big(\Phi(y_j^*)\big)_{j=1}^m \big\|_{p,\mathrm{weak}} : \\ m \in \mathbb{N},\ (y_j^*)_{j=1}^m \in (F^*)^m,\ \|(y_j^*)_{j=1}^m\|_{p,\mathrm{weak}} \leqslant 1 \Big\} < \infty.$$

  Then the restriction

$$C_\Phi\colon\ (H^p[E], \|\cdot\|_{\mathrm{FBL}^p[E]}) \to (H^p[F], \|\cdot\|_{\mathrm{FBL}^p[F]}), \quad f \mapsto f \circ \Phi,$$

  is bounded with norm $\|\Phi\|_p$.
- Suppose in addition that $\Phi\restriction_{B_{F^*}}$ is weak*-to-weak* continuous. Then

$$C_\Phi(H_{w^*}[E]) \subseteq H_{w^*}[F].$$

Question: Is $C_\Phi(I_{w^*}[E]) \subseteq \overline{I_{w^*}[F]}$?

Example. Let E and F be Banach spaces, where $E \neq \{0\}$ and F admits an ∞-dimensional, separable quotient space. Then there is a pos. homogeneous map $\Phi\colon F^* \to E^*$ with $\|\Phi\|_p < \infty$ and $\Phi\restriction_{B_{F^*}}$ weak*-to-weak* continuous, but

$$C_\Phi(\mathrm{FBL}^p[E]) \nsubseteq \mathrm{FBL}^p[F].$$

The representation theorem for lattice homomorphisms between FBLs

Theorem (L, Tradacete). Let $T\colon \mathrm{FBL}^p[E] \to \mathrm{FBL}^p[F]$ be a lattice homomorphism for some $1 \leqslant p < \infty$ and some Banach spaces E and F. Then there is a unique map $\Phi_T\colon F^* \to E^*$ such that

$$Tf = f \circ \Phi_T \qquad (f \in \mathrm{FBL}^p[E]).$$

It is positively homogeneous and satisfies:

- $\|\Phi_T\|_p = \|T\|$,
- $\Phi_T\restriction_{B_{F^*}}$ is weak*-to-weak* continuous.

Hope: This may help address the **isomorphism problem** for free Banach lattices:

Let E and F be Banach spaces and $1 \leqslant p < \infty$, and suppose that

$$\mathrm{FBL}^p[E] \cong \mathrm{FBL}^p[F] \quad \text{as Banach lattices.}$$

Is $E \cong F$ as Banach spaces?

The end — thank you!