Some recent results on mild Riesz* homomorphisms

Positivity XII Hammamet

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Riesz* homomorphisms

Let X, Y partially ordered vector spaces, $T\colon X \to Y$ linear.

Definition

• If X, Y vector lattices, T is a Riesz homomorphism if
$\forall a, b \in X\colon \quad T(a \vee b) = Ta \vee Tb.$

• (van Haandel 1993) T is a Riesz* homomorphism if
$\forall F \subseteq X$ finite nonempty: $\quad T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}}.$

• (Buskes–van Rooij 1993) T is a Riesz homomorphism if
$\forall a, b \in X\colon \quad T[\{a, b\}^{\mathrm{u}}]^{\mathrm{l}} = \{Ta, Tb\}^{\mathrm{ul}}.$

A^{u} = set of all upper bounds of A
A^{l} = set of all lower bounds of A
$A^{\mathrm{ul}} = (A^{\mathrm{u}})^{\mathrm{l}}.$

Riesz* homomorphisms

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Theorem (van Haandel 1993) Let E, F vector lattices, X, Y order dense linear subspace of E, F that generate E, F as vector lattices, resp. Let $T \colon X \to Y$ be linear. Then T extends to a lattice homomorphism $\hat{T} \colon E \to F$ if and only if T is a Riesz* homomorphism.

X order dense in E means $\forall y \in E\colon\ y = \inf\{x \in X \colon x \geq y\}$

X 'generates E as a vector lattice' means that the smallest Riesz subspace of E containing X is E itself.

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In terms of pre-Riesz spaces:

- A partially ordered vector space X is a pre-Riesz space if and only if there exists a vector lattice E and a bipositive linear $i\colon X \to E$ such that $i[X]$ is order dense in E and generates E as a vector lattice.
- Such an E is unique (up to isomorhpism of vector lattices) and called the Riesz completion of X.
- Each directed Archimedean pov is pre-Riesz.
- Riesz* homomorphisms are those maps between pre-Riesz spaces that extend to lattice homomorphisms between their Riesz completions.

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Theorem (van Haandel 1993) Let X be an order unit space, i.e., an Archimedean partially ordered vector space with an order unit u. Let $\Sigma = \{\varphi\colon X \to \mathbb{R}\colon\ \varphi$ positive and $\varphi(u) = 1\}$. Then $\varphi \in \Sigma$ is a Riesz* homomorphism if and only if $\varphi \in \overline{\mathrm{ext}\Sigma}$.

$\|x\| := \inf\{\lambda \in \mathbb{R}\colon\ -\lambda u \leq x \leq \lambda u\}$ order unit norm on X, weak* topology on norm dual of X.

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Question Is it sufficient to consider $F = \{a, b\}$ in the definition of Riesz* homomorphism?

No! (Boisen, Hölker, Kalauch, Stennder, vG, 2024)

mild Riesz* homomorphisms

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Boisen, Hölker, Kalauch, Stennder, vG, 2024:

Let X, Y partially ordered vector spaces, $T \colon X \to Y$ linear.

Definition T is a mild Riesz* homomorphism if
$\forall a, b \in X\colon \quad T[\{a, b\}^{\mathrm{ul}}] \subseteq \{Ta, Tb\}^{\mathrm{ul}}$.

Theorem Let X be a finite dimensional order unit space with a generating polyhedral cone and $\varphi \colon X \to \mathbb{R}$ linear. Then
φ is a mild Riesz* homomorphism $\iff$ φ is a Riesz* homomorphism.

mild Riesz* homomorphisms

Boisen, Hölker, Kalauch, Stennder, vG, 2024:

Let X be an Archimedean partially ordered vector space with an order unit u. Let $\varphi\colon X \to \mathbb{R}$ be a linear functional.

- φ is a Riesz* homomorphism if $\forall F \subseteq X$ finite nonempty: $\varphi[F^{\mathrm{ul}}] \subseteq \varphi[F]^{\mathrm{ul}}$.
- φ is a mild Riesz* homomorphism if $\forall a, b \in X$: $\varphi[\{a, b\}^{\mathrm{ul}}] \subseteq \{\varphi(a), \varphi(b)\}^{\mathrm{ul}}$.

$\Sigma := \{\phi\colon X \to \mathbb{R}\colon\ \phi \text{ is positive and } \phi(u) = 1\}$

Theorem Assume X is 3-dimensional.

- If Σ is strictly convex, then φ is a mild Riesz* homomorphism $\iff$ φ is positive.
- If Σ is not strictly convex then φ is a mild Riesz* homomorphism $\iff$ φ is a Riesz* homomorphism.

Exa: ice cream cone in $\mathbb{R}^3$, Σ is a disk, every $\varphi \in \Sigma \setminus \mathrm{ext}(\Sigma)$ is mild Riesz* homomorphism but not Riesz* homomorphism.

mild Riesz* homomorphisms of degree n

After Boisen, Hölker, Kalauch, Stennder, vG, 2024:
Mainly work by Florian Boisen. Some by Prashand Rambaran and vG.

Let X, Y partially ordered vector spaces, $T\colon X \to Y$ linear.

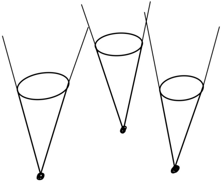
Definition Let $n \in \mathbb{N}$. T is a mild Riesz* homomorphism of degree n if $\forall F \subseteq X$ with $1 \leq |F| \leq n$: $T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}}$.
T is an n-mild Riesz* homomorphism

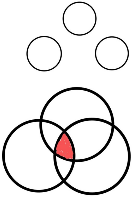
- If $n = 1$: T is positive
- If $n = 2$: T is a mild Riesz* homomorphism
- T mild Riesz* hom. of degree n for all $n \iff T$ Riesz* hom.
- T mild Riesz* hom. of degree $n \implies T$ mild Riesz* hom. of degree k for all $k \leq n$

A geometric approach

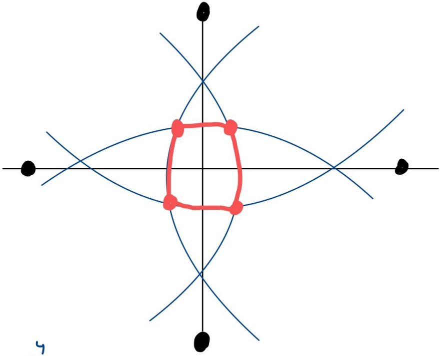
with Prashand Rambaran

K closed cone in $\mathbb{R}^3$ such that $(0,0,1) \in \operatorname{int} K$.
$\varphi((x,y,z)) := z$. Let $n \in \mathbb{N}$.
φ mild Riesz* homomorphism of degree n: for all $F = \{x_1, \ldots, x_n\}$:
$\varphi[F^{\mathrm{ul}}] \subseteq \varphi[F]^{\mathrm{ul}}$ $(*)$
Write $x_i = (x_i(1), x_i(2), x_i(3))$.
RHS of $(*)$: $\varphi[F] = \{x_1(3), \ldots, x_n(3)\}$, so $\varphi[F]^{\mathrm{ul}} = \{\max_i x_i(3)\}^{\mathrm{l}}$.
LHS of $(*)$: $F^{\mathrm{u}} = (x_1 + K) \cap \cdots \cap (x_n + K)$,
so $v \in F^{\mathrm{ul}}$ if and only if $(x_1 + K) \cap \cdots \cap (x_n + K) \subseteq v + K$.
$(*)$ is most critical for v as high as possible.





A geometric approach



$\bigcap_{i=1}^{4} (x_i + K)$ at height $z > 1$

$K = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2, \ z \geq 0 \}$

ice cream cone

$\varphi((x,y,z)) = z$ is mild Riesz* hom of degree 2 as dish is strictly convex

$F := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$ (x_1, x_2, x_3, x_4)

$\varphi[F]^{ul} = (-\infty, 0]$.

- compute 'corners' of intersection
- show they are contained in dish with radius $z - \frac{1}{2}$.

So $\bigcap_{i=1}^{4} (x_i + K) \subseteq \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + K$ so $\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \in F^{ul}$

so $\frac{1}{2} \in \varphi[F^{ul}]$ so $\varphi[F^{ul}] \nsubseteq (-\infty, 0] = \varphi[F]^{ul}$

So φ not mild Riesz* of degree 4.

Description of mild Riesz* functionals

Florian Boisen

X Archimedean partially ordered vector space with order unit u,
$\Sigma := \{\varphi\colon X \to \mathbb{R}\colon\ \varphi \text{ is positive and } \varphi(u) = 1\}$
Functional representation: view $x \in X$ as a function on Σ (or $\overline{\mathrm{ext}(\Sigma)}$):
$\Phi(x)(\varphi) := \varphi(x)$, $\Phi\colon X \to \mathrm{C}(\Sigma)$, $\quad x \geq 0 \iff \Phi(x) \geq 0$.

Theorem Let $\sigma \in \Sigma$ and $n \in \mathbb{N}$. Equivalent are:
(a) σ is a mild Riesz* homomorphism of degree n, i.e. $\forall F \subseteq X$ with $1 \leq |F| \leq n$: $\varphi[F^{\mathrm{ul}}] \subseteq \varphi[F]^{\mathrm{ul}}$.
(b) $\forall x_1, \ldots, x_n \in X$ and $v \in X$:
$\Phi(v) \leq \bigvee_{i=1}^{n} \Phi(x_i)$ on $\overline{\mathrm{ext}(\Sigma)} \implies \Phi(v)(\sigma) \leq \bigvee_{i=1}^{n} \Phi(x_i)(\sigma)$.

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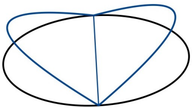

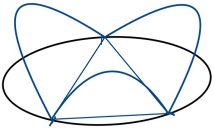

Description of mild Riesz* functionals

Theorem Let X be an order unit space of dimension $d \in \mathbb{N}$. Every mild Riesz* homomorphism of degree $n \geq d$ is a Riesz* homomorphism.

Conclusion

- $\forall n \geq m$:
Riesz* homomorphism $\Longrightarrow$ mild Riesz* homomorphism of degree n $\Longrightarrow$ mild Riesz* homomorphism of degree m
- In order unit spaces of dimension d: $\forall n \geq d$
Riesz* homomorphisms $=$ mild Riesz* homomorphisms of degree n
- There exists a mild Riesz* homomorphism of degree 2 which is not a mild Riesz* homomorphism of degree 3.

Description of mild Riesz* functionals

Theorem Let X be an order unit space of dimension $d \in \mathbb{N}$. Every mild Riesz* homomorphism of degree $n \geq d$ is a Riesz* homomorphism.

Conclusion

- $\forall n \geq m$:
Riesz* homomorphism $\Longrightarrow$ mild Riesz* homomorphism of degree n $\Longrightarrow$ mild Riesz* homomorphism of degree m
- In order unit spaces of dimension d: $\forall n \geq d$
Riesz* homomorphisms = mild Riesz* homomorphisms of degree n
- There exists a mild Riesz* homomorphism of degree 2 which is not a mild Riesz* homomorphism of degree 3.

THANK YOU! MERCI!