Positive Operators and Dimension Spectrum of Continued Fraction expansion

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Continued Fractions Expansion

The continued fraction expansion of an irrational number $x \in [0,1]$ is given by:

$$x = [a_1, a_2, a_3, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

for some $a_i \in \mathbb{N}$ and $i \in \mathbb{N}$.

Example:

$$\sqrt{2} - 1 = [2, 2, 2, \cdots] = \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2 + \cdots}}}.$$

It is also a fixed point in $[0,1]$ of the map $\frac{1}{2+x}$.

Hausdorff Dimension

Given $A \subseteq \mathbb{N}$, denote by J_A the set of all irrational numbers whose continued expansion digits belong to A.

$$J_A = \{x \in (0,1) \colon x = [a_1, a_2, a_3 \cdots] \text{ with } a_i \in A \text{ for all } i\},$$

These sets have a fractal pattern and their Hausdorff dimension, $\dim_{\mathcal{H}}(J_A)$, have been studied extensively.
For example:

- $\dim_{\mathcal{H}}(J_{\mathbb{N}}) = 1$ and
- $\dim_{\mathcal{H}}(J_{\{1,2\}}) \approx 0.531280506343388$.
- $\dim_{\mathcal{H}}(J_{\{2^n \colon n \in \mathbb{N}\}}) \in (0.4720715327, 0.4720715331)$

Invariant set

J_A is the invariant set of the **Iterated Function System** $\{\theta_a \colon a \in A\}$ where $\theta_a \colon [0,1] \longrightarrow [0,1]$ defined by

$$\theta_a(x) = \frac{1}{a+x},\ x \in [0,1].$$

The set J_A satisfies

$$J_A = \bigcup_{a \in A} \theta_a(J_A).$$

Dimension Spectrum

Given $A \subseteq \mathbb{N}$, define the **dimension spectrum** of A as,

$$\mathcal{DS}(A) = \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq A\}.$$

Q : What is the structure of $\mathcal{DS}(A)$ for different choices of A.

- $0 \in \mathcal{DS}(A)$ as $\dim_{\mathcal{H}}(J_F) = 0$ if $|F| = 1$ and $\dim_{\mathcal{H}}(J_A) \in \mathcal{DS}(A)$.
- If A is finite, then $\mathcal{DS}(A)$ is finite.
- If A is infinite, the structure is not well understood, but it is a closed and perfect subset of the interval $[0, 1]$.

Texan Conjecture : $\mathcal{DS}(\mathbb{N}) = [0, 1]$, Kesserbohmer and Zhu, (2006).

The work of Chousionis, Leykekhman and Urbanksi

Recently, $\mathcal{DS}(A)$ of infinite subsets of $\mathbb{N}$ was studied by Chousionis, Leykekhman and Urbanksi (*TAMS, 2019*) for various $A \subseteq \mathbb{N}$.

They showed the **fullness** of $\mathcal{DS}(A)$, i.e.,

$$\mathcal{DS}(A) = [0, \dim_{\mathcal{H}}(J_A)]$$

for :

- Arithmetic progression $A_{q,m} = \{q + mn \colon n \in \mathbb{N}\}$ for some $m, q \in \mathbb{N}$,
- $A_{prime} = \{p \colon p \text{ is prime }\}$,
- $M_2 = \{n^2 \colon n \in \mathbb{N}\}$,

Q : Is $\mathcal{DS}(M_q)$ full for $M_q = \{n^q \colon n \in \mathbb{N}\}$ for all $q \in \mathbb{N}$

Set of Powers:

For $P_q = \{q^n : n \in \mathbb{N}\}$ with $q \geq 2$. They showed that there exists $s(q) > 0$ such that

$$[0, \max\{s(q), \dim_{\mathcal{H}}(J_{P_q})\}] \subset \mathcal{DS}(A)$$

Question

- Is $\mathcal{DS}(P_q)$ full?
  Note: $1 \notin P_q$

We give a positive answer

Result 1: The set $M_q = \{n^q : \in \mathbb{N}\}$

Theorem : The dimension spectrum of M_q satisfies

- $\mathcal{DS}(M_q)$ is a disjoint union of finitely many nontrivial closed intervals.
- For $1 \leq q \leq 5$, $\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q})]$, i.e. it is full.
- For $q \geq 6$ we have $\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q\}})$ and $\mathcal{DS}(M_q) \cap (\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}), \dim_{\mathcal{H}}(J_{\{1,2^q\}}))$ is empty.
- For $6 \leq q \leq 8$,

$$\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q})]$$

The set $M_q = \{n^q : \in \mathbb{N}\}$

Theorem : The dimension spectrum of M_q satisfies

- For $q \in \{9, 10, 11\}$ we have that $\dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ and

$$\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q})].$$

Q : Is there an apriori upper bound for the number of intervals? Can we determine the $q's$ where the number of intervals changes?

Result 2: Sets with sub-multiplicative property

Theorem : If $A = \{a_1, a_2, \dots\} \subset \mathbb{N}$ with $2 \leq a_1 < a_2 < \dots$ and $a_{n+m} \leq a_n a_m$ for all $n, m \in \mathbb{N}$, then

$$\mathcal{DS}(A) = [0, \dim_{\mathcal{H}}(J_A)].$$

- In particular P_q is full. It also implies several results by Chousionis, Leykekhman and Urbanksi.
- For instance, $A_{\text{even}} = 2\mathbb{N}$ has a full dimension spectrum. All sets with an arithmetic progression and $a_1 \geq 2$ are full.
- Using the fact that the n-th prime p_n satisfies

  $$n(\ln n + \ln\ln n - 1) < p_n < n(\ln n + \ln\ln n) \quad \text{for } n \geq 6,$$

  it can be shown that $p_{n+m} \leq p_n p_m$ for all $n, m \geq 1$, hence $A_{\text{primes}} = \{p_1, p_2, \dots\}$ also has full dimension spectrum.

Result 3 : The set $P_q^* = P_q \cup \{1\}$

The dimension spectrum of

$$P_q^* = P_q \cup \{1\}, \ \ q \geq 2$$

has a much more complicated structure. More specifically, given $q \geq 2$ and $k \geq 0$. Let

$$F_k = \{1, \ldots, q^k\}$$

and for $k \geq 1$ set

$$\sigma^k = \dim_{\mathcal{H}}(J_{P_q^* \setminus \{q^k\}}) \quad \text{and} \quad \tau^k = \dim_{\mathcal{H}}(J_{F_k}).$$

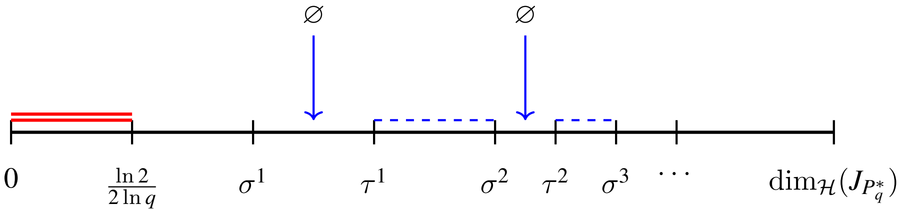
We have the following result.
Theorem For all $q \geq 3$ and $k \geq 1$,

(i) $\sigma^k < \tau^k$ and $(\sigma^k, \tau^k) \cap \mathcal{DS}(P_q^*)$ is empty.

(ii) $\mathcal{DS}(P_q^*)$ is nowhere dense in (τ^k, σ^{k+1}).

For $q = 2$, assertions (i) and (ii) hold for all $k \geq 2$.

Result 4: Non -trivial Interval



Theorem : For $q \in \mathbb{N}$ with $q \geq 2$ we have that the interval $[0, \frac{\ln 2}{2\ln q}]$ is contained in $\mathcal{DS}(P_q^*)$, and $\mathcal{DS}(P_q^*)$ is nowhere dense in

$$\left(\sigma^1, \dim_{\mathcal{H}}(J_{P_q^*})\right].$$

The Perron-Frobenius Operator

Given $F \subset \mathbb{N}$ finite and $s \geq 0$, define $L_{s,F} \colon C([0,1]) \longrightarrow C([0,1])$ by:

$$(L_{s,F}f)(x) = \sum_{a \in F} |\theta_a'(x)|^s f(\theta_a(x)) = \sum_{a \in F} \left(\frac{1}{a+x}\right)^{2s} f\left(\frac{1}{a+x}\right).$$

This defines a positive, bounded linear operator on $C([0,1])$.

For $s > 0$, let

$$K_s = \left\{ f \in C([0,1]) \colon 0 \leq f(x) \leq f(y)e^{2s|x-y|}, \quad x, y \in [0,1] \right\}.$$

Then

- K_s is a closed cone in $C([0,1])$.
- $L_{s,F}(K_s) \subseteq K_s$.

LINK: Positive Operator and Hausdorff dimension

The following result is due to Falk and Nussbaum (*J. Fractal Geometry*, 2008)

Theorem : For all $F \subset \mathbb{N}$ finite and $s > 0$,

- The operator $L_{s,F}$ has a unique strictly positive eigenvector $v_s \in K_s$ with $L_{s,F}v_s = \lambda_s v_s$ where $\lambda_s > 0$ and $\lambda_s = r(L_{s,F})$.
- The spectrum $\sigma(L_{s,F}) \subset \mathbb{C}$ satisfies :

$$\sup\left\{\frac{|z|}{\lambda_{s,F}} : z \in \sigma(L_{s,F}) \setminus \{\lambda_{s,F}\}\right\} < 1.$$

- v_s is strictly decreasing on $[0,1]$ and $v_s \in K_s$.
- The map $s \mapsto \lambda_s$ is continuous and strictly decreasing.
- $\lambda_{s_0} = 1$ if and only if $\dim_{\mathcal{H}}(J_F) = s_0$.

Ideas in the proofs

We give an idea of the proof to have an understanding on how we use the spectral properties of $L_{s,F}$ to establish the result.

Let $F_1 = \{1, q\}$ and $P_q^* \setminus \{q\} = \{1, q^2, q^3, \ldots\}$

Claim :

$$\dim_{\mathcal{H}}(J_{P_q^* \setminus \{q\}}) = \sigma^1 < \tau^1 = \dim_{\mathcal{H}}(J_{F_1})$$

step 1 Find a lower bound of $\dim_{\mathcal{H}}(J_{F_1})$

step 2 Show that it works as an upper bound of $\dim_{\mathcal{H}}(J_{P_q^* \setminus \{q\}})$.

Special Perron-Frobenius Operator

Fact: If

$$(L_{s,\{1\}}f)(x) = \left(\frac{1}{1+x}\right)^{2s} f\left(\frac{1}{1+x}\right).$$

and we let $\mu = \frac{1+\sqrt{5}}{2}$ and $v_s(x) = \left(\frac{1}{\mu+x}\right)^{2s}$ then

$$(L_{s,\{1\}}v_s)(x) = \mu^{-2s} v_s(x).$$

Also, using this we get

$$\begin{aligned}(L_{s,F_1}v_s)(x) &= \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \left(\frac{1}{q+x}\right)^{2s} v_s\left(\frac{1}{q+x}\right) \\ &= \mu^{-2s}\left(1+\left(\frac{\mu+x}{q+x+\mu-1}\right)^{2s}\right) v_s(x).\end{aligned}$$

Idea of Proof

So

$$(L_{s,F_1} v_s)(x) \geq \mu^{-2s} \left(1 + \left(\frac{\mu}{q + \mu - 1} \right)^{2s} \right) v_s(x).$$

Now, if $s > 0$ is such that

$$\mu^{-2s} \left(1 + \left(\frac{\mu}{q + \mu - 1} \right)^{2s} \right) \geq 1$$

then $(L_{F_1,s} v_s)(x) \geq v_s(x)$.
As $L_{s,F}$ is a positive operator and v_s is strictly positive, hence $r(L_{F_1,s}) \geq 1$ so $\dim_{\mathcal{H}}(J_{F_1}) \geq s$.

Bounds For $\dim_{\mathcal{H}}(J_{F_1})$

Theorem : For $q \geq 4$, we have the following bound

$$\dim_{\mathcal{H}}(J_{F_1}) > \frac{0.52679}{\ln q} = s(q).$$

If we can show that $r(L_{s(q),P_q^*\setminus\{q\}}) < 1$. This implies that $\dim_{\mathcal{H}}(J_{P_q^*\setminus\{q\}}) = \sigma^1 < s(q) < \tau^1$ and that completes the proof.

$r(L_{s(q),P_q^*\setminus\{q\}}) < 1$

$L_{s,\{1,q\}}$ has eigenvector v_s with $r(L_{s,F}) = \lambda_s$ and

$$L_{s,P_q^*\setminus\{q\}} v_s(x) = \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \sum_{n\geq 2}\left(\frac{1}{q^n+x}\right)^{2s} v_s\left(\frac{1}{q^n+x}\right)$$

As $v_s \in K_s$ we have that

$$v_s\left(\frac{1}{q^n+x}\right) \leq v_s\left(\frac{1}{q+x}\right) e^{2s\left(\frac{1}{q+x}-\frac{1}{q^n+x}\right)} \leq v_s\left(\frac{1}{q+x}\right) e^{2s/q}.$$

Also

$$\frac{(q^n+x)^{-2s}}{(q+x)^{-2s}} = \left(\frac{q+x}{q^n+x}\right)^{2s} \leq \left(\frac{q+1}{q^n+1}\right)^{2s}.$$

$r(L_{s(q),P_q^*\setminus\{q\}}) < 1$

$$L_{s,P_q^*\setminus\{q\}}v_s(x) \leq \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \frac{e^{4s/q}}{q^{2s}-1}\left(\frac{1}{q+x}\right)^{2s} v_s\left(\frac{1}{q+x}\right)$$

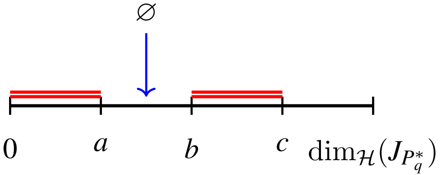
One can show that if $s = s(q)$, then $\frac{e^{4s/q}}{q^{2s}-1} < 1$ so there exists a $\mu < 1$ such that

$$L_{s,P_q^*\setminus\{q\}}v_s(x) \leq \mu v_s(x)$$

Future work

Given $A \subseteq \mathbb{N}$ infinite and $0 < a < b < c < d$.

1. If $[a, b] \subset \mathcal{DS}(A)$ and $[c, d] \subset \mathcal{DS}(A)$ and $\mathcal{DS}(A)$ is nowhere dense in (b, c) then we should have that $\mathcal{DS}(A) \cap (b, c)$ is empty.



2. if $\mathcal{DS}(A)$ contains a solid interval $[a, b]$ with $a > 0$ then there exists a $\delta > 0$ such that $[0, \delta] \subset \mathcal{DS}(A)$.

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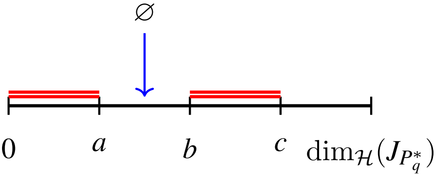

2. if $\mathcal{DS}(A)$ contains a solid interval $[a, b]$ with $a > 0$ then there exists a $\delta > 0$ such that $[0, \delta] \subset \mathcal{DS}(A)$.

[1] P. Chitanga, B. Lemmens and R.D Nussbaum, On the structure of the dimension spectrum for continued fraction expansion. *arXiv:2504.20878* , (2025).