Generalized Inverses of Disjointness Preserving Operators on Pre-Riesz Spaces

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Joint work with:
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Inverse preserver problem

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Q.2 If T preserves a relation R, does some generalized inverse T^G preserve the relation?

<u>Generalized Inverses:</u>

If $T \in \mathcal{B}(H_1, H_2)$ has closed range, then there exists a unique $U \in \mathcal{B}(H_2, H_1)$ such that

$$TUT = T,\ UTU = U,\ (TU)^* = TU,\ (UT)^* = UT. \tag{†}$$

This unique operator U is called the Moore-Penrose inverse[1] of T. It is denoted by $T^\dagger$.

[1] [Ben-Israel and Greville, 2003] Generalized inverses. Theory and applications.
[Groetsch, 1977] Generalized inverses of linear operators. Representation and approximation.

Least squares solutions

Consider

$$T(x) = y$$

where $y \in H_2$ and $T \in \mathcal{B}(H_1, H_2)$.
If $\mathsf{R}(T)$ is closed, then $\exists!\ u \in \mathsf{R}(T)$ such that

$$\|u - y\| \leq \|w - y\|, \quad \forall w \in \mathsf{R}(T).$$

Consider

$$\mathcal{A} := T^{-1}\{u\}.$$

$\mathcal{A}$ is a closed convex set in a Hilbert space and, hence, contains a unique element of least norm, say $\hat{u}$.

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Then, we have

$$T^{\dagger} y = \hat{u}.$$

Drazin inverse

Let X be a vector space and $T\colon X \to X$ a linear operator. A linear operator $S\colon X \to X$ is called the Drazin inverse[2] of T, denoted by T^D, if, for some $k \in \mathbb{Z}_+$,

$$T^k ST = T^k, \quad STS = S, \quad TS = ST.$$

[2] [Campbell and Meyer, 2009] Generalized inverses of linear transformations .

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- If $k \leq 1$, then S is called the group inverse of T, denoted by $T^{\#}$.

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- If $k \leq 1$, then S is called the group inverse of T, denoted by $T^{\#}$.
- If X is finite dimensional, then T^D always exists.
- Let T^D and $T^\dagger$ both exist. Then $T^\dagger = T^D$ if and only if $\mathsf{R}(T) = \mathsf{R}(T^*)$.

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Application of group inverses in finite Markov chains

For an m-state Markov chain whose one-step transition matrix is T, consider the matrix $A = I - T$. As Carl D. Meyer said[3]:

"For an ergodic chain, virtually everything that one would want to know about the chain can be determined by computing $A^{\#}$."

[3] [Meyer, 1975] The role of the group generalized inverse in the theory of finite Markov chains.

Partially ordered vector spaces:

Questions

Special partially ordered vector spaces:

- Riesz spaces (vector lattices).
- Pre-Riesz spaces.

Structure preserving linear maps:

- Riesz homomorphism.
- Riesz* homomorphism.
- Disjointness preserving maps (d.p.).

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Q.1 Let X be a partially ordered vector space and $T\colon X \to X$ be a structure preserving linear map. If T is bijective, is then T^{-1} of the same type?

Q.2 If T is not bijective but T^D or $T^\dagger$ exist, is then T^D or $T^\dagger$ of the same type?

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  X is pre-Riesz space and T a Riesz* hom. $\not\Rightarrow$ T^{-1} is Riesz* hom.

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- [2019, Kalauch, Lemmens and Gaans]

  X Archimedean finite dimensional pre-Riesz space and $T\colon X \to X$ d.p. $\implies T^{-1}$ d.p.

Riesz homomorphism

Let X, Y be Riesz spaces and $T\colon X \to Y$ a linear operator. Then T is called a Riesz homomorphism if, for every $x, y \in X$, one has

$$T(x \vee y) = T(x) \vee T(y).$$

Proposition (Kalauch, R., Sivakumar, 2025)

Let X be a Riesz space and $T\colon X \to X$ a Riesz homomorphism. If T^D exists, then T^D is a Riesz homomorphism.

Pre-Riesz spaces and Riesz* homomorphisms:

Definition (pre-Riesz space):
A partially ordered vector space X is called a pre-Riesz space[4] if

$$X \xrightarrow[\text{bipositive}]{\Phi} Y \text{ (Riesz space)}, \qquad \Phi[X] \overset{\text{order dense}}{\subseteq} Y.$$

Such a Riesz space Y is called a vector lattice cover for X.

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Definition (Riesz completion):
If, for every $y \in Y$,

$$y = \bigvee_{i=1}^{n} \Phi(a_i) - \bigvee_{j=1}^{m} \Phi(b_j),$$

then (Y, Φ) is called the Riesz completion of X, denoted by X^ρ.

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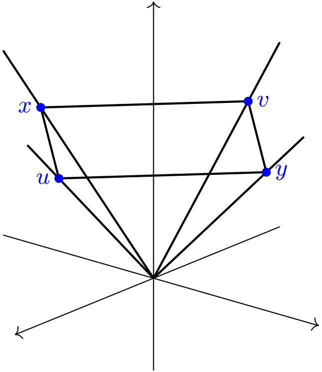

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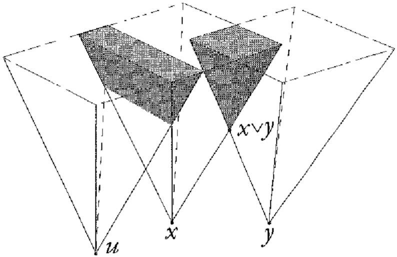

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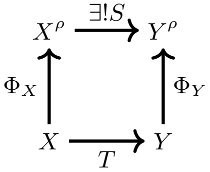
Let X, Y be pre-Riesz spaces. A linear operator $T\colon X \to Y$ is called a Riesz* homomorphism if there exists a Riesz homomorhism $S\colon X^\rho \to Y^\rho$ between their Riesz completions that extends T.



$$S \circ \Phi_X = \Phi_Y \circ T.$$

The extension S is uniquely determined by T, and is called the van Haandel extension of T.

Bijective Riesz* homomorphism

Q. Let X be a pre-Riesz space and $T\colon X \to X$ a Riesz* homomorphism. If T is bijective, is then T^{-1} a Riesz* homomorphism?

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Example ([van Imhoff, 2018])

Let X be the space of all polynomials on $[0,1]$ with the point-wise order. Let $T\colon X \to X$ be the linear map defined as

$$Tp(x) = p(\tfrac{1}{2}x) \quad \forall x \in [0,1],\ p \in X.$$

It is shown that T is a bijective Riesz* homomorphism and T^{-1} is not a Riesz* homomorphism.

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The Riesz completion X^ρ is the space of all continuous piecewise polynomial functions on $[0,1]$. The van Haandel extension $S\colon X^\rho \to X^\rho$ is given as

$$Sf(x) = f(\tfrac{1}{2}x) \quad \forall x \in [0,1],\ f \in X^\rho.$$

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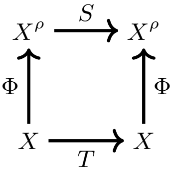
Observe that

$$\ker S = \{f \in X^\rho \colon \forall x \in [0,\tfrac{1}{2}];\ f(x) = 0\}.$$

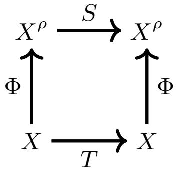
Hence S is not bijective.

A characterization



Theorem (Kalauch, R., Sivakumar, 2025)

Let X be a pre-Riesz space and X^ρ the Riesz completion of X. Let $T\colon X \to X$ be a Riesz homomorphism and $S\colon X^\rho \to X^\rho$ the van Haandel extension of T. Let T^D exist. Then T^D is a Riesz* homomorphism if and only if S^D exists.*



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Corollary (Kalauch, R., Sivakumar, 2025)

If T is bijective, then $T^{-1}\colon X \to X$ is a Riesz homomorphism if and only if S is bijective.*

Polyhedral cones in $\mathbb{R}^n$

- Let $K \subseteq \mathbb{R}^n$ be a generating polyhedral cone in $\mathbb{R}^n$.
- Then $(\mathbb{R}^n, K)$ is a pre-Riesz space, and $\mathbb{R}^k$ with the point-wise order is the Riesz completion[5] of $(\mathbb{R}^n, K)$ for a suitable k.

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Theorem (Kalauch, R., Sivakumar, 2024)

If $T\colon (\mathbb{R}^n, K) \to (\mathbb{R}^n, K)$ is a Riesz homomorphism, then so is T^D.*

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Disjointness:

- For a vector lattice X and $x, y \in X$, we say that x and y are disjoint, denoted by $x \perp y$, if

$$|x| \wedge |y| = 0.$$

Example:
Let Ω be a compact Hausdorff space and $\mathsf{C}(\Omega)$ the space of all real valued continuous functions on Ω with the point-wise order. For any $f, g \in \mathsf{C}(\Omega)$ and $x \in \Omega$,

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- Let X be a pre-Riesz space and Y a Riesz cover of X. For $x, y \in X$, we say that x and y are disjoint, denoted by $x \perp y$, if

$$\Phi(x) \perp \Phi(y).$$

Disjoint elements in the 4-ray cone

Define

$$K := \text{pos}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3.$$

$(\mathbb{R}^3, K)$ is a pre-Riesz space, referred to as the 4-ray cone, with the Riesz completion $(\mathbb{R}^4, \Phi)$, where

$$\Phi \colon \mathbb{R}^3 \to \mathbb{R}^4, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix}.$$

Thus, for $x, y \in \mathbb{R}^3$, we have

$$x \perp y \iff \forall i \colon f_i(x) f_i(y) = 0$$
$$\iff x \in U \text{ and } y \in V,$$

$$\text{span}\{(1,0,1)\} \perp \text{span}\{(1,0,-1)\}$$
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Disjointness preserving operators

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Theorem (Kalauch, Lemmens, van Gaans, 2014)

If X is a finite dimensional Archimedean pre-Riesz space and T is a bijective disjointness preserving operator, then so is T^{-1}.

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Theorem (Kalauch, R., Sivakumar, 2025)

Let X be a finite dimensional Archimedean pre-Riesz space. If $T\colon X \to X$ is disjointness preserving operator, then so is T^{D}.

Summary

Is $T^D : X \to X$ of the same type?

Space (X)	Operator (T)	Answer
Riesz space	Riesz homomorphism	✓
pre-Riesz spaces	Riesz*	$\checkmark \Leftrightarrow S^D$ exists
Archimedean finite dimensional pre-Riesz spaces	d.p.	✓
Banach lattice	d.p.	Open

Remark

One cannot expect the analogous results for the Moore-Penrose inverse.

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Thank you all for your attention :)