Commutators greater than a perturbation of the identity

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Theorem (Brown and Pearcy, 1964)

Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. A bounded operator C on $\mathscr{H}$ is a commutator if and only if it is not of the form $\lambda I + K$ for some nonzero scalar λ and some compact operator K on $\mathscr{H}$.

Let $C \geq I$ be any bounded operator on the Hilbert lattice ℓ^2 which is not of the form $\lambda I + K$ for some scalar λ and some compact operator K. Then by the above theorem there exist bounded operators A and B on ℓ^2 such that $[A, B] := AB - BA = C \geq I$. One may ask whether A and B can be also positive operators on the Hilbert lattice ℓ^2. The answer to this question is negative, as we have the following theorem that is inspired by Wielandt's proof of the Wintner-Wielandt result.

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Theorem (Drnovšek, Kandić, 2025)

Let A and B be bounded operators on a Banach lattice. If $[A, B] \geq I$, then neither A nor B is positive.

Proof.

Suppose first that A is positive. By induction, for each $n \in \mathbb{N}$ we have $[A^n, B] \geq nA^{n-1}$. We conclude that $n\|A^{n-1}\| \leq \|[A^n, B]\|$, and the submultiplicativity of the norm gives us

$$n\|A^{n-1}\| \leq \|A^n B - BA^n\| \leq 2\|A\| \, \|B\| \, \|A^{n-1}\|.$$

Now, if $A^k = 0$ for some $k \in \mathbb{N}$, then the inequality $0 = [A^k, B] \geq kA^{k-1} \geq 0$ yields $A^{k-1} = 0$. It follows that $A^n \neq 0$ for any $n \in \mathbb{N}$. We obtain that, for each $n \in \mathbb{N}$, $n \leq 2\|A\| \, \|B\|$. This contradiction shows that A is not positive.
If B is positive, we rewrite $[A, B] = [B, -A]$ and apply the first part of the proof. □

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If B is positive, we rewrite $[A,B] = [B,-A]$ and apply the first part of the proof. □

We have also the next quantitative version of the last theorem.

Theorem (Drnovšek, Kandić, 2025)

Let A and B be bounded operators on a Banach lattice. Suppose that at least one of them is positive, and that for some $\varepsilon > 0$ there exists an operator E with $\|E\| \le \varepsilon$ such that

$$[A, B] \ge I + E.$$

Then

$$\|A\| \cdot \|B\| \ge \frac{1}{2} \ln \frac{1}{\varepsilon}.$$

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We now give an example of such operators.

Let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of the Hilbert lattice ℓ^2. The operator $U\colon \ell^2 \to \ell^2$ which is defined on the standard basis vectors as $Ue_n = e_{2n}$ ($n \in \mathbb{N}$) is a positive isometry. Similarly, the operator $V\colon \ell^2 \to \ell^2$ which is defined on the standard basis vectors as $Ve_n = e_{2n-1}$ ($n \in \mathbb{N}$) is also a positive isometry. Hence, U and V can be realized as infinite matrices

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with respect to the standard basis $(e_n)_{n\in\mathbb{N}}$. Then we have $UU^* + VV^* = I$, $U^*U = V^*V = I$ and $U^*V = V^*U = 0$.

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Using U and V we can prove the following theorem.

Theorem (Drnovšek, Kandić, 2025)

There exist positive operators $A, B\colon \ell^2 \to \ell^2$ such that $[A, B] = I + N$, where N is a nilpotent operator of nil-index 3. Furthermore, if $\varepsilon \in (0,1)$, then A and B can be chosen in such a way that $\|A\| = O(\varepsilon^{-3})$, $\|B\| = O(\varepsilon^{-3})$ and $\|N\| = O(\varepsilon)$.

Of course, it is not possible that the nilpotent operator N in the theorem is positive. Let us sketch the proof of the theorem. Let $W := UV^* + VU^*$. Then W satisfies $We_{2n} = e_{2n-1}$ and $We_{2n-1} = e_{2n}$ for each $n \in \mathbb{N}$. Therefore,

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Now we define 4×4 block-operator matrices

$$A = \begin{pmatrix} 0 & \varepsilon V^* & 0 & 3\varepsilon^3 I \\ 0 & U^* & \varepsilon I & 0 \\ \frac{1}{\varepsilon^2} V^* & 0 & U^* & 2\varepsilon W \\ \frac{1}{\varepsilon^3} U^* & 0 & \frac{1}{\varepsilon} V^* & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 2\varepsilon^2 V & 2\varepsilon^3 U \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\varepsilon} I & 2U & 2\varepsilon V \\ \frac{1}{\varepsilon^3} I & 0 & 0 & 0 \end{pmatrix}$$

that define positive operators on $\ell^2 \cong \ell^2 \oplus \ell^2 \oplus \ell^2 \oplus \ell^2$ satisfying $\|A\| = O(\varepsilon^{-3})$ and $\|B\| = O(\varepsilon^{-3})$.

A direct calculation now yields

$$[A,B] = \begin{pmatrix} I & 0 & -2\varepsilon^2 W & -4\varepsilon^3 VW \\ 0 & I & 2\varepsilon U & 2\varepsilon^2 V \\ 0 & 0 & I & -4\varepsilon UW \\ 0 & 0 & 0 & I \end{pmatrix},$$

that can be written as $I + N$, where

$$N = \begin{pmatrix} 0 & 0 & -2\varepsilon^2 W & -4\varepsilon^3 VW \\ 0 & 0 & 2\varepsilon U & 2\varepsilon^2 V \\ 0 & 0 & 0 & -4\varepsilon UW \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\|N\| = O(\varepsilon)$.

In the finite-dimensional setting, the situation is quite different, as the notion of the trace restricts drastically which operators are commutators. An $n \times n$ matrix over an arbitrary field is a commutator if and only if its trace is zero.

The order analog of this classical result is the following.

Theorem (Drnovšek, Kandić, 2019)

A non-negative matrix C can be written as a commutator of non-negative matrices A and B if and only if C is nilpotent. Moreover, we can choose A to be diagonal and B to be permutation similar to a strictly upper-triangular matrix.

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Proposition

Let A and B be real $n \times n$ matrices. Assume that there exists a real $n \times n$ matrix X such that

$$[A, B] \geq I - X.$$

Then $\mathrm{tr}(X) \geq n$ and $r(X) \geq 1$, and so $\|X\| \geq 1$.

For more related results one can consult the paper:

R. Drnovšek, M. Kandić: *Commutators greater than a perturbation of the identity*, *J. Math. Anal. Appl.* 541 (2025) 128713.