Institute of Analysis, **Leibniz University Hannover**

Positivity in infinite-dimensional time-invariant systems

Positivity XII, Tunisia

June 3, 2025

Sahiba Arora

Motivation

PDE

For $s \in [0,1]$, $\tau \geq 0$, consider

$$x_\tau(\tau, s) = x_s(\tau, s),$$
$$x(0, s) = x_0(s),$$
$$x(\tau, 1) = 0.$$

Motivation

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$x_\tau(\tau,s) = x_s(\tau,s),$$
$$x(0,s) = x_0(s),$$
$$x(\tau,1) = 0.$$

Abstract Cauchy problem (ACP)

On $X = \{f \in C[0,1] : f(1) = 0\}$,

$$\dot{x}(\tau) = Ax(\tau), x(0) = x_0;$$

Motivation

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$x_\tau(\tau, s) = x_s(\tau, s),$$
$$x(0, s) = x_0(s),$$
$$x(\tau, 1) = 0.$$

Abstract Cauchy problem (ACP)

On $X = \{f \in C[0,1] : f(1) = 0\}$,

$$\dot{x}(\tau) = Ax(\tau), x(0) = x_0;$$

where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}$$
$$Af := f'.$$

Motivation

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$\begin{aligned} x_\tau(\tau,s) &= x_s(\tau,s), \\ x(0,s) &= x_0(s), \\ x(\tau,1) &= 0. \end{aligned}$$

Abstract Cauchy problem (ACP)

On $X = \{f \in C[0,1] : f(1) = 0\}$,

$$\dot{x}(\tau) = Ax(\tau), x(0) = x_0;$$

where

$$\begin{aligned} D(A) &= \{f \in C^1[0,1] : f(1) = f'(1) = 0\} \\ Af &:= f'. \end{aligned}$$

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$\begin{aligned} x_\tau(\tau,s) &= x_s(\tau,s) + \int_0^1 x(\tau,r)\, dr \cdot h(s), \\ x(0,s) &= x_0(s), \\ x(\tau,1) &= 0; \end{aligned}$$

where $h \in L^1(0,1)_+$.

Motivation

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$x_\tau(\tau, s) = x_s(\tau, s),$$
$$x(0, s) = x_0(s),$$
$$x(\tau, 1) = 0.$$

Abstract Cauchy problem (ACP)

On $X = \{f \in C[0,1] : f(1) = 0\}$,

$$\dot{x}(\tau) = Ax(\tau), x(0) = x_0;$$

where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}$$
$$Af := f'.$$

PDE

For $s \in [0,1], \tau \geq 0$, consider

$$x_\tau(\tau, s) = x_s(\tau, s) + \int_0^1 x(\tau, r)\, dr \cdot h(s),$$
$$x(0, s) = x_0(s),$$
$$x(\tau, 1) = 0;$$

where $h \in L^1(0,1)_+$.

Linear time-invariant (LTI) system

$$\dot{x}(\tau) = Ax(\tau) + Bx(\tau), x(0) = x_0;$$

with

$$B : X \to \left\{ \begin{matrix} g \in \mathcal{D}(0,1)' : g = \partial f \\ \text{for some } f \in X \end{matrix} \right\}$$
$$x \mapsto \left(\int_0^1 x(r)\, dr \right) \cdot h.$$

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \tag{ACP}$$

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if

$T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if $T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if

$T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.

Solution to (ACP): $x(t) = T(t)x_0$.

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if
$T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.
Solution to (ACP): $x(t) = T(t)x_0$.

Each semigroup is associated to a unique generator:

$$D\left(A\right) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \qquad Ax := \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if $T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.
Solution to (ACP): $x(t) = T(t)x_0$.

Each semigroup is associated to a unique generator:

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \qquad Ax := \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

Extrapolation space

X_{-1} : completion of X with $\|x\|_{-1} := \left\| (\lambda - A)^{-1} x \right\|$ for fixed $\lambda \in \rho(A)$.

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if

$T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.

Solution to (ACP): $x(t) = T(t)x_0$.

Each semigroup is associated to a unique generator:

$$D\left(A\right) = \left\{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\right\}, \qquad Ax := \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

Extrapolation space

X_{-1} : completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Example: For $X = \{f \in C[0,1] : f(1) = 0\}$ and

$$D\left(A\right) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f',$$

$X_{-1} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some } f \in X\}$[1].

[1] Batkai, Jacob, Voigt, & Wintermayr

Zooming-in on the abstract Cauchy problem

For a closed operator A on a Banach space X, consider

$$\dot{x}(\tau) = Ax(\tau),\ x(0) = x_0. \qquad \text{(ACP)}$$

Fact: (ACP) has a mild solution if and only if A generates a C_0-semigroup on X.

Definition

A family of bounded operators $(T(t))_{t\geq 0}$ is called a C_0-semigroup if $T(0) = \mathrm{id}$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and $t \mapsto T(t)x$ is continuous for each x.

Solution to (ACP): $x(t) = T(t)x_0$.

Each semigroup is associated to a unique generator:

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \qquad Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t}.$$

Extrapolation space

X_{-1} : completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Example: For $X = \{f \in C[0,1] : f(1) = 0\}$ and

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f',$$

$X_{-1} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some } f \in X\}$[1].

Fact: $(T(t))_{t\geq 0}$ extends to C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} extends A.

[1] Batkai, Jacob, Voigt, & Wintermayr

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

$A :$ generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in \mathcal{L}(U, X_{-1})$ (Banach space U)

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in \mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\text{loc}}(\mathbb{R}_+, U)$, we say B is Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in \mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\text{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\,ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\mathrm{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\,ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$,

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\text{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$, heat equation with Dirichlet boundary control and $Z = L^\infty$.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\mathrm{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$, heat equation with Dirichlet boundary control and $Z = L^\infty$.

Properties: (a) L^1-admissible $\xRightarrow{p\in(1,\infty)}$ L^p-admissible $\Rightarrow$ L^∞-admissible $\Rightarrow$ C-admissible.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\mathrm{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$, heat equation with Dirichlet boundary control and $Z = L^\infty$.

Properties: (a) L^1-admissible $\xRightarrow{p\in(1,\infty)}$ L^p-admissible $\Rightarrow$ L^∞-admissible $\Rightarrow$ C-admissible.

(b) If $U = X$, then zero-class C-admissibility $\Rightarrow$ $A_{-1} + B$ generates C_0-semigroup on X.

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in$ $\mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\mathrm{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0, \tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$, heat equation with Dirichlet boundary control and $Z = L^\infty$.

Properties: (a) L^1-admissible $\xrightarrow{p\in(1,\infty)}$ L^p-admissible $\Rightarrow$ L^∞-admissible $\Rightarrow$ C-admissible.

(b) If $U = X$, then zero-class C-admissibility $\Rightarrow$ $A_{-1} + B$ generates C_0-semigroup on X.

How does positivity help in obtaining automatic admissibility?

Zooming-on the linear time-invariant (control) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau) + Bu(\tau), \quad \tau \geq 0$$
$$x(0) = x_0$$

Fact: solutions have the form
$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B \in \mathcal{L}(U, X_{-1})$ (Banach space U)

X_{-1} : Completion of X with $\|x\|_{-1} := \left\|(\lambda - A)^{-1}x\right\|$ for fixed $\lambda \in \rho(A)$.

Definition

For $Z \subseteq L^1_{\text{loc}}(\mathbb{R}_+, U)$, we say B is zero-class Z-admissible if for some (hence, all) $\tau > 0$,

$$\Phi_\tau : Z([0,\tau], U) \to X_{-1}, \qquad u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

maps into X and if $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{Z([0,\tau],U)\to X} = 0$.

Examples: $B \in \mathcal{L}(U, X)$, heat equation with Dirichlet boundary control and $Z = L^\infty$.

Properties: (a) L^1-admissible $\xrightarrow{p\in(1,\infty)}$ L^p-admissible $\Rightarrow$ L^∞-admissible $\Rightarrow$ C-admissible.

(b) If $U = X$, then zero-class C-admissibility $\Rightarrow$ $A_{-1} + B$ generates C_0-semigroup on X.

How does positivity help in obtaining automatic admissibility?
$\rightarrow$ We first need to understand the order structure of X_{-1}!

Order on X_{-1}

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

Order on X_{-1}

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

On $X = \{f \in C[0,1] : f(1) = 0\}$, the PDE

$$\begin{aligned} x_\tau(\tau,s) &= x_s(\tau,s) && s \in [0,1], \tau \geq 0, \\ x(0,s) &= x(s) && s \in [0,1], \\ x(\tau,1) &= 0 && \tau \geq 0 \end{aligned}$$

is associated with the system $\dot{x}(\tau) = Ax(\tau), x(0) = x_0$, where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

On $X = \{f \in C[0,1] : f(1) = 0\}$, the PDE

$$\begin{aligned} x_\tau(\tau, s) &= x_s(\tau, s) & & s \in [0,1], \tau \geq 0, \\ x(0, s) &= x(s) & & s \in [0,1], \\ x(\tau, 1) &= 0 & & \tau \geq 0 \end{aligned}$$

is associated with the system $\dot{x}(\tau) = Ax(\tau), x(0) = x_0$, where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

The operator A generates a positive semigroup

Order on X_{-1}

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

On $X = \{f \in C[0,1] : f(1) = 0\}$, the PDE

$$\begin{aligned} x_\tau(\tau, s) &= x_s(\tau, s) && s \in [0,1], \tau \geq 0, \\ x(0, s) &= x(s) && s \in [0,1], \\ x(\tau, 1) &= 0 && \tau \geq 0 \end{aligned}$$

is associated with the system $\dot{x}(\tau) = Ax(\tau), x(0) = x_0$, where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

The operator A generates a positive semigroup and[1]

$$X_{-1} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some } f \in X\},$$
$$X_{-1,+} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some increasing } f \in X\}.$$

[1] Batkai, Jacob, Voigt, & Wintermayr

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

On $X = \{f \in C[0,1] : f(1) = 0\}$, the PDE

$$\begin{aligned} x_\tau(\tau, s) &= x_s(\tau, s) && s \in [0,1], \tau \geq 0, \\ x(0, s) &= x(s) && s \in [0,1], \\ x(\tau, 1) &= 0 && \tau \geq 0 \end{aligned}$$

is associated with the system $\dot{x}(\tau) = Ax(\tau), x(0) = x_0$, where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

The operator A generates a positive semigroup and[1]

$$X_{-1} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some } f \in X\},$$
$$X_{-1,+} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some increasing } f \in X\}.$$

In particular, $X_{-1} \supsetneq X_{-1,+} - X_{-1,+}$.

[1] Batkai, Jacob, Voigt, & Wintermayr

Let A generate a positive C_0-semigroup on a Banach lattice X. We define

$$X_{-1} \ni f \geq 0 :\Leftrightarrow f \in X_{-1,+} := \overline{\{f \in X : f \geq 0\}}^{\|\cdot\|_{-1}}.$$

On $X = \{f \in C[0,1] : f(1) = 0\}$, the PDE

$$\begin{aligned} x_\tau(\tau, s) &= x_s(\tau, s) && s \in [0,1], \tau \geq 0, \\ x(0, s) &= x(s) && s \in [0,1], \\ x(\tau, 1) &= 0 && \tau \geq 0 \end{aligned}$$

is associated with the system $\dot{x}(\tau) = Ax(\tau), x(0) = x_0$, where

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

The operator A generates a positive semigroup and[1]

$$X_{-1} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some } f \in X\},$$

$$X_{-1,+} = \{g \in \mathcal{D}(0,1)' : g = \partial f \text{ for some increasing } f \in X\}.$$

In particular, $X_{-1} \supsetneq X_{-1,+} - X_{-1,+}$.

In general, X_{-1} is a ordered Banach space with a normal cone but *not* a Banach lattice.

[1] Batkai, Jacob, Voigt, & Wintermayr

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

[1] A., Glück, Paunonen, & Schwenninger

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot\right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

[1] A., Glück, Paunonen, & Schwenninger

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot\right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

Order properties of $\operatorname{span} X_{-1,+}$

[1] A., Glück, Paunonen, & Schwenninger

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

Order properties of $\operatorname{span} X_{-1,+}$[2]

X has order continuous norm $\Rightarrow$ $\operatorname{span} X_{-1,+}$ is also a Banach lattice!

[1] A., Glück, Paunonen, & Schwenninger

[2] A., Glück, & Schwenninger

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot\right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

Order properties of $\operatorname{span} X_{-1,+}$[2]

X has order continuous norm $\Rightarrow$ $\operatorname{span} X_{-1,+}$ is also a Banach lattice!

Order properties of X preserved by $\operatorname{span} X_{-1,+}$: (a) Order continuous norm, (b) KB-space.

[1] A., Glück, Paunonen, & Schwenninger

[2] A., Glück, & Schwenninger

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

Order properties of $\operatorname{span} X_{-1,+}$[2]

X has order continuous norm $\Rightarrow \operatorname{span} X_{-1,+}$ is also a Banach lattice!

Order properties of X preserved by $\operatorname{span} X_{-1,+}$: (a) Order continuous norm, (b) KB-space.

Open question

Is $\operatorname{span} X_{-1,+}$ a Banach lattice if X doesn't have order continuous norm?

[1] A., Glück, Paunonen, & Schwenninger

[2] A., Glück, & Schwenninger

De-tour to order properties of X_{-1}

Let A: generate positive C_0-semigroup on X (Banach lattice) with $0 \in \rho(A)$,

$$X_{-1} := \left(X, \left\|A^{-1} \cdot \right\|\right)^{\sim} \quad \text{and} \quad X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}}$$

Order properties of X preserved by X_{-1}:[1]

(a) Norm-bounded increasing nets are norm-convergent, (b) Cone is a face of bidual wedge.

In case (b), X_+ is also a face of $X_{-1,+}$.

Order properties of $\operatorname{span} X_{-1,+}$[2]

X has order continuous norm $\Rightarrow$ $\operatorname{span} X_{-1,+}$ is also a Banach lattice!

Order properties of X preserved by $\operatorname{span} X_{-1,+}$: (a) Order continuous norm, (b) KB-space.

Open question

Is $\operatorname{span} X_{-1,+}$ a Banach lattice if X doesn't have order continuous norm?

Some related results[2]

Let $k \in \mathbb{N}$ and $p \in (1, \infty)$. Then $\operatorname{span} W^{-k,p}(\Omega)_+$ is a KB-space if $\Omega = \mathbb{R}^d$ or (a, b).

---

[1] A., Glück, Paunonen, & Schwenninger

[2] A., Glück, & Schwenninger

Automatic zero-class $\mathbb{C}$-admissibility

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s) B u(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U) \to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t \geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\} \subseteq \rho(A)$

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

B is called zero-class $\mathbb{C}$-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\,ds$$

satisfies $\lim_{\tau\downarrow 0}\|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))$

i.e., B maps the unit ball of U into an order bounded subset of X_{-1}

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Automatic zero-class $\mathbb{C}$-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s) B u(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U) \to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t \geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t \geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t \geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau], U)$[1] with $\|u\|_\infty \leq 1$,

$$\int_0^\tau T_{-1}(\tau - s) b_1\, ds \leq \Phi_\tau u \leq \int_0^\tau T_{-1}(\tau - s) b_2\, ds$$

[1] Tent (step) functions

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau],U)$[1] with $\|u\|_\infty \leq 1$,

$$\underbrace{\int_0^\tau T_{-1}(\tau - s)b_1\, ds}_{\in D(A_{-1})=X} \leq \Phi_\tau u \leq \underbrace{\int_0^\tau T_{-1}(\tau - s)b_2\, ds}_{\in D(A_{-1})=X}.$$

[1] Tent (step) functions

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

$A :$ generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

$B : U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau],U)$[1] with $\|u\|_\infty \leq 1$,

$$\underbrace{\int_0^\tau T_{-1}(\tau - s)b_1\, ds}_{\in D(A_{-1})=X} \leq \Phi_\tau u \leq \underbrace{\int_0^\tau T_{-1}(\tau - s)b_2\, ds}_{\in D(A_{-1})=X}.$$

Therefore, $\Phi_\tau u \in X$

[1] Tent (step) functions

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau],U)$[1] with $\|u\|_\infty \leq 1$,

$$\underbrace{\int_0^\tau T_{-1}(\tau - s)b_1\, ds}_{\in D(A_{-1})=X} \leq \Phi_\tau u \leq \underbrace{\int_0^\tau T_{-1}(\tau - s)b_2\, ds}_{\in D(A_{-1})=X}.$$

Therefore, $\Phi_\tau u \in X$ and by normality of cone $\exists\, c > 0$ such that

$$\|\Phi_\tau u\|_X \leq c \max_{i=1,2} \left\| \int_0^\tau T_{-1}(\tau - s)b_i\, ds \right\|_X$$

[1] Tent (step) functions

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau], U)$[1] with $\|u\|_\infty \leq 1$,

$$\underbrace{\int_0^\tau T_{-1}(\tau - s)b_1\, ds}_{\in D(A_{-1})=X} \leq \Phi_\tau u \leq \underbrace{\int_0^\tau T_{-1}(\tau - s)b_2\, ds}_{\in D(A_{-1})=X}.$$

Therefore, $\Phi_\tau u \in X$ and by normality of cone $\exists\, c > 0$ such that

$$\|\Phi_\tau u\|_X \leq c \max_{i=1,2} \left\| A_{-1} \int_0^\tau T_{-1}(\tau - s) b_i\, ds \right\|_{-1}$$

[1] Tent (step) functions

Automatic zero-class $\mathbb{C}$-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Proof (outline): For each $u \in T([0,\tau],U)$[1] with $\|u\|_\infty \leq 1$,

$$\underbrace{\int_0^\tau T_{-1}(\tau - s)b_1\, ds}_{\in D(A_{-1})=X} \leq \Phi_\tau u \leq \underbrace{\int_0^\tau T_{-1}(\tau - s)b_2\, ds}_{\in D(A_{-1})=X}.$$

Therefore, $\Phi_\tau u \in X$ and by normality of cone $\exists\, c > 0$ such that

$$\|\Phi_\tau u\|_X \leq c \max_{i=1,2} \left\| A_{-1} \int_0^\tau T_{-1}(\tau - s)b_i\, ds \right\|_{-1} = c \max_{i=1,2} \|T_{-1}(\tau)b_i - b_i\|_{-1} \xrightarrow{\tau\downarrow 0} 0. \qquad \square$$

[1] Tent (step) functions

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau \downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let μ: finite continuous positive Borel measure on $(0,1)$ and $h \in L^1(0,1)$.

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\,ds$$

satisfies $\lim_{\tau\downarrow 0}\|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let μ: finite continuous positive Borel measure on $(0,1)$ and $h \in L^1(0,1)$. Then

$$f \mapsto B_1 f := \int_0^1 f(s)\,ds \cdot \mu \quad \& \quad f \mapsto B_2 f := hf$$

lie in $\mathcal{L}(X, X_{-1})$,

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let μ: finite continuous positive Borel measure on $(0,1)$ and $h \in L^1(0,1)$. Then

$$f \mapsto B_1 f := \int_0^1 f(s)\, ds \cdot \mu \quad \& \quad f \mapsto B_2 f := hf$$

lie in $\mathcal{L}(X, X_{-1})$, $B_1(B_X) \subseteq [-\mu, \mu]$ & $B_2(B_X) \subseteq [-|h|, |h|]$.

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{C([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let μ: finite continuous positive Borel measure on $(0,1)$ and $h \in L^1(0,1)$. Then

$$f \mapsto B_1 f := \int_0^1 f(s)\, ds \cdot \mu \quad \& \quad f \mapsto B_2 f := hf$$

lie in $\mathcal{L}(X, X_{-1})$, $B_1(B_X) \subseteq [-\mu, \mu]$ & $B_2(B_X) \subseteq [-|h|, |h|]$.
Thus, B_i is zero-class C-admissible

Automatic zero-class C-admissibility

B is called zero-class C-admissible if

$$\Phi_\tau : u \mapsto \int_0^\tau T_{-1}(\tau - s)Bu(s)\, ds$$

satisfies $\lim_{\tau\downarrow 0} \|\Phi_\tau\|_{\mathrm{C}([0,\tau],U)\to X} = 0$.

A : generator of C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

B : $U \xrightarrow{\text{bounded}} X_{-1}$ (Banach space U)

WLOG : $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0\} \subseteq \rho(A)$

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

Sufficient: $U = X$ is AM-space, $(T(t))_{t\geq 0}$ and B are positive, and $\mathrm{r}(A^{-1}B) < 1$.

Theorem (A., Glück, Paunonen, & Schwenninger)

Sufficient: X is a Banach lattice, $(T(t))_{t\geq 0}$ is positive, and $B(B_U) \subseteq [b_1, b_2]$.

Example

On $X = \{f \in C[0,1] : f(1) = 0\}$, consider

$$D(A) = \{f \in C^1[0,1] : f(1) = f'(1) = 0\}, \qquad Af := f'.$$

Let μ: finite continuous positive Borel measure on $(0,1)$ and $h \in L^1(0,1)$. Then

$$f \mapsto B_1 f := \int_0^1 f(s)\, ds \cdot \mu \quad \& \quad f \mapsto B_2 f := hf$$

lie in $\mathcal{L}(X, X_{-1})$, $B_1(B_X) \subseteq [-\mu, \mu]$ & $B_2(B_X) \subseteq [-|h|, |h|]$.
Thus, B_i is zero-class C-admissible $\Rightarrow$ $A_{-1} + B_i$ generates C_0-semigroup on X.

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

Definition

C is called L^1-admissible if

$$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \qquad x \mapsto CT(\cdot)x$$

has a bounded extension to X for some (hence, all) $\tau > 0$.

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

Definition

C is called L^1-admissible if

$$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \qquad x \mapsto CT(\cdot)x$$

has a bounded extension to X for some (hence, all) $\tau > 0$.

Example: $C \in \mathcal{L}(X, Y)$.

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

Definition

C is called L^1-admissible if

$$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \qquad x \mapsto CT(\cdot)x$$

has a bounded extension to X for some (hence, all) $\tau > 0$.

Example: $C \in \mathcal{L}(X, Y)$.
Example: Heat equation on $L^2[0,1]$ with Dirichlet BC and point observation.

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

Definition

C is called L^1-admissible if

$$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \qquad x \mapsto CT(\cdot)x$$

has a bounded extension to X for some (hence, all) $\tau > 0$.

Example: $C \in \mathcal{L}(X, Y)$.
Example: Heat equation on $L^2[0,1]$ with Dirichlet BC and point observation.
Non-example: $X = Y$: reflexive & $C = A$ with $D(A) \neq X$.[1]

[1] Jacob, Schwenninger, & Wintermayr

Linear time-invariant (observation) system

Consider the linear time-invariant system

$$\dot{x}(\tau) = Ax(\tau), \quad x(0) = x_0$$
$$y(\tau) = Cx(\tau).$$

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Fact: $(T(t))_{t\geq 0}$ leaves $D(A)$ invariant.

Definition

C is called L^1-admissible if

$$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \qquad x \mapsto CT(\,\cdot\,)x$$

has a bounded extension to X for some (hence, all) $\tau > 0$.

Example: $C \in \mathcal{L}(X, Y)$.
Example: Heat equation on $L^2[0,1]$ with Dirichlet BC and point observation.
Non-example: $X = Y$: reflexive & $C = A$ with $D(A) \neq X$.[1]

How does positivity help in obtaining automatic admissibility?

[1] Jacob, Schwenninger, & Wintermayr

When does positivity imply L^1-admissibility?

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

[1] X_+: generating and normal

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau],Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline):

[1] X_+: generating and normal

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \; x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline): Tweaking Wintermayr's proof, we can weaken the order properties to

[1] X_+: generating and normal

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline): Tweaking Wintermayr's proof, we can weaken the order properties to

X, Y: ordered Banach spaces[1] such that norm on Y is additive on Y_+.

[1] X_+: generating and normal

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline): Tweaking Wintermayr's proof, we can weaken the order properties to

X, Y: ordered Banach spaces[1] such that norm on Y is additive on Y_+.

Finite rank $\Rightarrow$ $\exists$ equivalent norm on $\operatorname{Rg} C$ additive on $(\operatorname{Rg} C)_+$.

[1] X_+: generating and normal

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline): Tweaking Wintermayr's proof, we can weaken the order properties to

X, Y: ordered Banach spaces[1] such that norm on Y is additive on Y_+.

Finite rank $\Rightarrow$ $\exists$ equivalent norm on $\operatorname{Rg} C$ additive on $(\operatorname{Rg} C)_+$. So,

$$\|\Psi_\tau x\|_{L^1([0,\tau],Y)} \leq \|\mathrm{id}\|_{\operatorname{Rg} C \to Y} \|\Psi_\tau\|_{X \to L^1([0,\tau],\operatorname{Rg} C)} \|x\|_X \qquad \forall\, x \in D(A).$$

[1] X_+: generating and normal

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y), \; x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Proof (outline): Tweaking Wintermayr's proof, we can weaken the order properties to

X, Y: ordered Banach spaces[1] such that norm on Y is additive on Y_+.

Finite rank $\Rightarrow$ $\exists$ equivalent norm on $\operatorname{Rg} C$ additive on $(\operatorname{Rg} C)_+$. So,

$$\|\Psi_\tau x\|_{L^1([0,\tau],Y)} \leq \|\mathrm{id}\|_{\operatorname{Rg} C \to Y} \|\Psi_\tau\|_{X \to L^1([0,\tau],\operatorname{Rg} C)} \|x\|_X \qquad \forall\, x \in D(A).$$

Hence, C is L^1-admissible.

[1] X_+: generating and normal

When does positivity imply L^1-admissibility?

C is called L^1-admissible if
$\Psi_\tau : D(A) \to L^1([0,\tau], Y),\ x \mapsto CT(\cdot)x$
extends boundedly to X for all $\tau > 0$.

A : generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Banach space X

C : $D(A) \xrightarrow{\text{bounded}} Y$ (Banach space)

Theorem (Wintermayr)

Sufficient: X: Banach lattice, Y: AL-space, and $(T(t))_{t\geq 0}$ & C: positive.

Proposition (A., Glück, Paunonen, Schwenninger)

Sufficient: X, Y: ordered Banach spaces[1], $(T(t))_{t\geq 0}$ & C: positive, and C: finite rank.

Theorem (A., Glück, Paunonen, Schwenninger)

Sufficient: $X, \widetilde{Y}$: ordered Banach spaces[1], norm additive on $\widetilde{Y}_+$, $(T(t))_{t\geq 0}$: positive, and

$$C : D(A) \xrightarrow[\text{positive}]{C_1} \widetilde{Y} \xrightarrow[\text{bounded}]{C_2} Y.$$

[1] X_+: generating and normal

Systems with control and observation

Let X: Banach lattice, U: AM-space, and suppose

$A:$ generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Let X: Banach lattice, U: AM-space, and suppose

$A:$ generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D\left(A_B\right) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Systems with control and observation

Let X: Banach lattice, U: AM-space, and suppose

A : generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Let X: Banach lattice, U: AM-space, and suppose

A: generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Application: Let A_m: differential operator with maximal domain on a Banach lattice X,

Let X: Banach lattice, U: AM-space, and suppose

A : generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Application: Let A_m: differential operator with maximal domain on a Banach lattice X, $G : D(A_m) \xrightarrow{\text{onto}} \partial X$ (AM-space with unit) such that

Let X: Banach lattice, U: AM-space, and suppose

A : generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Application: Let A_m: differential operator with maximal domain on a Banach lattice X,
$G : D(A_m) \xrightarrow{\text{onto}} \partial X$ (AM-space with unit) such that
$\exists\, \lambda : \left(G|_{\ker(\lambda - A_m)} \right)^{-1} \in \mathcal{L}(\partial X, X)_+,$

Systems with control and observation

Let X: Banach lattice, U: AM-space, and suppose

A : generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B : U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Application: Let A_m: differential operator with maximal domain on a Banach lattice X, $G : D(A_m) \xrightarrow{\text{onto}} \partial X$ (AM-space with unit) such that $\exists\, \lambda : \left(G|_{\ker(\lambda - A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X)_+$, and $\Phi : X \xrightarrow{\text{positive}} \partial X$.

Systems with control and observation

Let X: Banach lattice, U: AM-space, and suppose

$A:$ generator of positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B: U \xrightarrow{\text{positive}} X_{-1}$.

Wlog, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$.

Theorem (Batkai, Jacob, Voigt, & Wintermayr)

If $U = X$ and $\mathrm{r}(A^{-1}B) < 1$, then

$$D(A_B) := \{f \in X : (A_{-1} + B)f \in X\}, \qquad A_B := (A_{-1} + B)|_X$$

generates a positive C_0-semigroup on X.

Theorem (Barbieri & Engel)

If $C : X \xrightarrow{\text{positive}} U$ and $\mathrm{r}(CA^{-1}B) < 1$, then

$$D(A_{BC}) := \{f \in X : (A_{-1} + BC)f \in X\}, \qquad A_{BC} := (A_{-1} + BC)|_X$$

generates a positive C_0-semigroup on X.

Application: Let A_m: differential operator with maximal domain on a Banach lattice X, $G : D(A_m) \xrightarrow{\text{onto}} \partial X$ (AM-space with unit) such that $\exists\, \lambda : \left(G|_{\ker(\lambda - A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X)_+$, and $\Phi : X \xrightarrow{\text{positive}} \partial X$. If $A_m|_{\ker G}$ generates a positive semigroup on X, then so does $A_m|_{\ker(G-\Phi)}$.

Selected references for infinite-dimensional positive systems

- S. Arora, J. Glück, L. Paunonen, and F. L. Schwenninger. **Limit-case admissibility for positive infinite-dimensional systems.** *J. Differ. Equations*, 440(1), 2025.
- S. Arora, J. Glück, and F. L. Schwenninger. **The lattice structure of negative Sobolev and extrapolation spaces.** *Isr. J. Math. (to appear)*, 2025.
- A. Barbieri and K.-J. Engel. **Perturbations of positive semigroups factorized via AM- and AL-spaces.** *J. Evol. Equ.*, 25(1), 2025.
- A. Bátkai, B. Jacob, J. Voigt, and J. Wintermayr. **Perturbations of positive semigroups on AM-spaces.** *Semigroup Forum*, 96(2), 2018.
- A. Boulouz, H. Bounit, and S. Hadd. **Feedback theory approach to positivity and stability of evolution equations.** *Syst. Control Lett.*, 161, 2022.
- Y. E. Gantouh. **Boundary approximate controllability under positivity constraints of infinite-dimensional control systems.** *J. Optim. Theory Appl.*, 198(2), 2023.
- Y. E. Gantouh. **Positivity of infinite-dimensional linear systems.** 2023. Preprint.
- Y. E. Gantouh. **Well-posedness and stability of a class of linear systems.** *Positivity*, 28(2), 2024.
- J. Wintermayr. ***Positivity in perturbation theory and infinite-dimensional systems.*** PhD thesis, Bergische Universität Wuppertal, 2019.

BERGISCHE UNIVERSITÄT WUPPERTAL

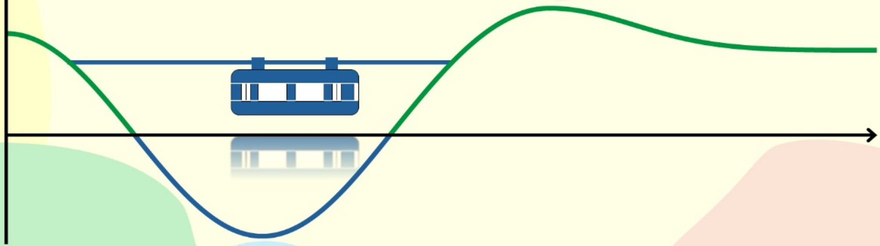
Leibniz Universität Hannover

ISem29

Eventual Positivity

Lecturers

Sahiba Arora
Jochen Glück
Jonathan Mui



Lectures

Oct 2025 –
Feb 2026

Projects

Feb 2026 –
Jun 2026

Workshop

8th – 12th Jun 2026
Wuppertal