Ortho-compact and d-compact operators

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In the sequel all operators are linear, $\mathcal{H}$ denotes a real or complex Hilbert space, X and Y Banach spaces, and E and F Banach lattices. By B_X we denote a closed unit ball in X; by $\mathrm{L}(X, Y)$ the space of bounded operators from X to Y, by $\mathrm{K}(X, Y)$ the space of compact operators, etc.

It is shown in [S.G., On compact (limited) operators between Hilbert and Banach spaces, *Filomat* (2024)]:

An operator $T : \mathcal{H} \to Y$ is bounded (compact, weakly compact, limited)

$$\Updownarrow$$

T takes **orthonormal sequences** of $\mathcal{H}$ into bounded (compact, weakly compact, limited) subset of Y

The following fact is well known [e.g., P.R. Halmos, *A Hilbert Space Problem Book*, 1982; p. 95 and p. 293]

$$T \in K(\mathcal{H}_1, \mathcal{H}_2)$$

$$\Updownarrow$$

$\|Tx_n\| \to 0$ for every orthonormal sequence (x_n) in $\mathcal{H}_1$

This fact was extended to operators from a Hilbert space to Banach space in [S.G., *Filomat* (2024)].

We give the following orthonormal characterization of compactness for operators (Theorem 2.3 in [S.G., *Filomat* (2024)]).

Theorem 1

Let $T : \mathcal{H} \to Y$. TFAE.

i) T is compact.

ii) T maps every orthonormal sequence of $\mathcal{H}$ to a norm-null sequence.

iii) T maps every orthonormal basis of $\mathcal{H}$ into a relatively compact set.

In *iii*) "every" cannot be replaced by "some" [Halmos (1982), p.292].

We have (see, Lemma 1.2 in [S.G., *Filomat* (2024)])

Proposition 1

An operator $T : \mathcal{H} \to Y$ is bounded

$$\Updownarrow$$

(Tx_n) is bounded for every orthonormal sequence (x_n) in $\mathcal{H}$.

The weakly compact version of Theorem 1 follows immediately from Proposition 1, because every bounded $T : \mathcal{H} \to Y$ is weakly compact.

Theorem 2

Let $T : \mathcal{H} \to Y$. TFAE.

i) T is weakly compact.

ii) T maps every orthonormal sequence of $\mathcal{H}$ to a w-null sequence.

iii) T maps every orthonormal basis of $\mathcal{H}$ into a relatively weakly compact set.

iv) T maps some orthonormal basis of $\mathcal{H}$ into a relatively weakly compact set.

Recall that a subset $G \subset X$ is **limited** if, for every (f_n) in X',

$$f_n \xrightarrow{w^*} 0 \quad \Longrightarrow \quad f_n \rightrightarrows 0(G).$$

And that $T : X \to Y$ is **limited** if $T(B_X)$ is limited in Y.

Since each limited set is bounded, Proposition 1 implies

Proposition 2

An operator $T : \mathcal{H} \to Y$ is bounded

$$\Updownarrow$$

the sequence (Tx_n) is limited for every orthonormal sequence (x_n) in $\mathcal{H}$.

Obviously, $\mathrm{K}(\mathcal{H}, Y) \subseteq \mathrm{Lim}(\mathcal{H}, Y)$. In general, the inclusion is proper.

Example 1

Let A be an ortonormal basis in $\mathcal{H}$, $\dim(\mathcal{H}) = \infty$.

Let $Y = \ell^{\infty}(A)$ be the space of bounded scalar functions on A.

Define $T : \mathcal{H} \to Y$ as follows:

$$(Tx)(u) = (x, u) \quad x \in \mathcal{H}, \quad u \in A$$

T is not compact (since $(Tu)_u = 1$, all others are zero).

As A is an orthonormal basis, $T(B_{\mathcal{H}}) \subseteq B_{c_0(A)}$.

Since $B_{c_0(A)}$ is limited in Y, T is limited.

Next, we give orthonormal characterization of limited operators (see Theorem 2.6 in [S.G., *Filomat* (2024)])

Theorem 3

Let $T : \mathcal{H} \to Y$ be linear and $\dim(\mathcal{H}) = \infty$. TFAE.

i) T is limited.

ii) T maps every orthonormal basis of $\mathcal{H}$ onto a limited set.

iii) T maps every orthonormal sequence of $\mathcal{H}$ onto a limited set.

In *ii*), "every" cannot be replaced by "some".

Indeed, limited sets in $\mathcal{H}$ agree with relatively compact sets, and hence $\mathrm{K}(\mathcal{H}) = \mathrm{Lim}(\mathcal{H})$. So, apply again the above example of P. Halmos.

Note that, the orthonormal bases and sequences in above theorems and propositions can be replaced by orthogonal bounded sequences.

The situation is less pleasant for operators acting from a Banach lattice to a Banach space.

Surprisingly, operators carrying disjoint bounded sequences of E into (weakly) compact (limited) subsets of Y behave differently with (weakly) compact (limited) operators. That is, we have several new classes of operators [E. Emelyanov, N. Erkursun-Özcan, S.G., d-Operators in Banach Lattices, arxiv.org/abs/2401.08792 (2024)].

We will use **disjoint bounded** ($:= \mathrm{dbdd}$) sequences. Such sequences play an important role in the Banach lattice theory. Our idea is to use the following kind of definition of d-operators.

T is d-xxx if T carries dbdd sequences into xxx sets.

(Here xxx is bounded, (weakly) compact, limited, etc.)

Open Question 1

Is every d-bounded operator bounded?

Open Question 2

Is every d-compact operator bounded?

Since we do not know answers to Questions 1 and 2, in what follows, we assume all operators to be bounded.

Let us consider compactness. Clearly

$$\boxed{\mathrm{Mwc}(E, Y) \subseteq \mathrm{d\text{-}K}(E, Y) \text{ and } \mathrm{K}(E, Y) \subseteq \mathrm{d\text{-}K}(E, Y),}$$

where $\mathrm{Mwc}(E, Y)$ is the space of M-weakly compact operators.

We are going to show that both of these inclusions are proper in general.

Let $T : \ell^1 \to \ell^\infty$ be defined by

$$Ta = \left(\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} a_k, \dots \right) \quad (a = (a_k)_k \in \ell^1).$$

$T \in \mathrm{K}(\ell^1, \ell^\infty)$ (it has rank one), and hence $\boxed{T \in \text{d-K}(\ell^1, \ell^\infty).}$ However, $\boxed{T \notin \mathrm{Mwc}(\ell^1, \ell^\infty)}$, since, for the dbdd-sequence (e_n) of standard unit vectors of ℓ_1, $\|Te_n\| \equiv 1$.

The following example shows that a d-compact operator need not be compact (Example 1 in [E. Emelyanov, N. Erkursun-Özcan, S.G., d-Operators in Banach Lattices. (2024)]).

Let $T : L^\infty[0,1] \to L^p[0,1]$ be the natural embedding $(1 \leq p < \infty)$.

Clearly $\boxed{T \notin \mathrm{K}(L^\infty[0,1], L^p[0,1]) \text{ because}}$

$$\|T(r_n^+) - T(r_m^+)\|_p = \|r_n^+ - r_m^+\|_p = 2^{-\frac{2}{p}} \quad r \neq m$$

(r_n is the n-th Rademacher function on $[0,1]$).

However, $\boxed{T \in \mathrm{d\text{-}K}(L^\infty[0,1], L^p[0,1]).}$

Indeed, $(Tf_n) = (f_n)$ is $\|\cdot\|_p$-null for each dbdd-sequence (f_n) in $L^\infty[0,1]$.

As every infinite-dimensional normed lattice E contains a normalized disjoint sequence, I_E is not d-compact. It follows

$$\boxed{\mathrm{d\text{-}K}(E) = \mathrm{L}(E) \iff \dim(E) < \infty.}$$

Obviously, $\boxed{\mathrm{W}(E, Y) \subseteq \mathrm{d}\text{-}\mathrm{W}(E, Y).}$

A d-weakly compact operator can be neither weakly compact nor d-compact (Example 2 in [E. Emelyanov, N. Erkursun-Özcan, S.G., d-Operators in Banach Lattices. (2024)]).

To see this, consider an operator

$$T : C[0,1] \to c : \quad Tx = \left(x\left(\frac{1}{k}\right)\right)_{k=1}^{\infty}.$$

We are going to show

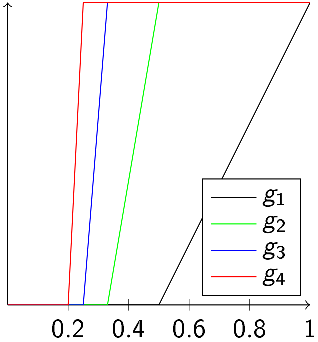
$$\boxed{T \in \mathrm{d\text{-}W}(C[0,1], c) \setminus (\mathrm{W}(C[0,1], c) \cup \mathrm{d\text{-}K}(C[0,1], c))}.$$

First, suppose (x_n) is a dbbd sequence in $C[0,1]$.

Since $(C[0,1])'$ is a KB-space, $x_n \xrightarrow{w} 0$, [1] and hence $Tx_n \xrightarrow{w} 0$ (by continuity), as desired. Thus, $\boxed{T \in \mathrm{d\text{-}W}(C[0,1], c).}$

[1] Aliprantis and Burkinshow, Positive operators. (2006) Theorem 4.69

Take $g_n \in C[0,1]$ as follows



Since

$$Tx = \left(x\left(\frac{1}{k}\right)\right)_{k=1}^{\infty}$$

and

$$g_n(t) = \begin{cases} 0 & \text{if} \quad t \leq \frac{1}{n+1}, \\ 1 & \text{if} \quad t \geq \frac{1}{n}, \\ & \text{is linear otherwise,} \end{cases}$$

we obtain

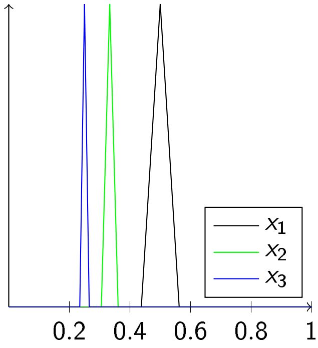
$Tg_1 = (1, 0, 0, \ldots),$

$Tg_2 = (1, 1, 0, \ldots), \ldots$

$Tg_n = (1, 1, \ldots, 1, 0, \ldots).$

Since (Tg_n) has no w-convergent subsequence then $\boxed{T \notin \mathrm{W}(C[0,1], c).}$

take $x_n \in C[0,1]$ as follows



As

$$Tx = \left(x\left(\frac{1}{k}\right)\right)_{k=1}^{\infty},$$

for the d-bdd sequence (x_n)

$$x_n(t) = \begin{cases} 0 & \text{if} \quad \left|t - \frac{1}{n+1}\right| > \frac{1}{4(n+1)^2}, \\ 1 & \text{if} \quad t = \frac{1}{n+1}, \\ & \text{is linear otherwise.} \end{cases}$$

we have $Tx_1 = (0,1,0,\dots)$,
$Tx_2 = (0,0,1,0,\dots),\dots,$
$Tx_n = e_{n+1} \in c$ for each $n \in \mathbb{N}$.

Since $T(\{x_n\}_{n=1}^{\infty}) = \{e_k : k \in \mathbb{N}\}$ is not relatively compact in c then

$$\boxed{T \notin \text{d-K}(C[0,1], c).}$$

Clearly, $\boxed{\mathrm{Lim}(E, Y) \subseteq \mathrm{d\text{-}Lim}(E, Y) \text{ and } \mathrm{d\text{-}K}(E, Y) \subseteq \mathrm{d\text{-}Lim}(E, Y)}$

Let us show that both inclusions are proper in general.

It was shown above (p. 16) that the natural embedding $T : L^\infty[0,1] \to L^1[0,1]$ is d-compact, and hence

$$\boxed{T \in \mathrm{d\text{-}Lim}(L^\infty[0,1], L^1[0,1])}$$

since relatively compact sets are limited.

As $L^1[0,1]$ is separable, its limited sets agree with relatively compact sets.

So, $\boxed{T \notin \mathrm{Lim}(L^\infty[0,1], L^1[0,1])}$ because T is not compact.

A limited operator needs not to be d-compact, and a w-compact operator needs not to be d-limited. (Examples 3 and 4 in [E. Emelyanov, N. Erkursun-Özcan, S.G., d-Operators in Banach Lattices. (2024)]).

Let $T : c_0 \to \ell^\infty$ be the embedding operator. Since (by Phillip's lemma) $T(B_{c_0}) = B_{c_0}$ is limited in ℓ^∞ then $\boxed{T \in \mathrm{Lim}(c_0, \ell^\infty).}$

Since $(Te_n) = (e_n)$ has no convergent subsequence, then $\boxed{T \notin \mathrm{d\text{-}K}(c_0, \ell^\infty).}$

As ℓ^2 is reflexive (so, B_{ℓ^2} is w-compact in ℓ^2) and the set $\{e_n : n \in \mathbb{N}\}$ is not limited in ℓ^2 then $\boxed{I_{\ell^2} \in \mathrm{W}(\ell^2) \setminus \mathrm{d\text{-}Lim}(\ell^2).}$

We have no example of $T \in \mathrm{d\text{-}Lim}(E, Y) \setminus \mathrm{W}(E, Y)$.

Obviously, we have

$$d\text{-}\mathrm{Lim}(E,Y)$$

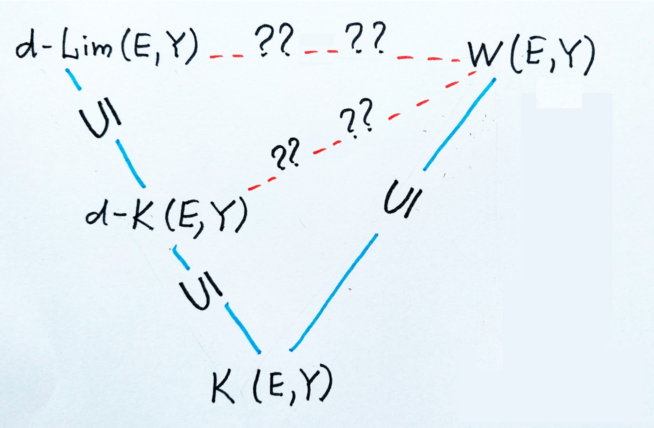
$$\cup\!|$$

$$d\text{-}K(E,Y)$$

$$K(E,Y) \subseteq d\text{-}K(E,Y) \supseteq Mwc(E,Y)$$

$$K(E,Y) \subseteq d\text{-}K(E,Y) \subseteq d\text{-}\mathrm{Lim}(E,Y), \qquad Mwc(E,Y) \subseteq d\text{-}K(E,Y)$$

However, there are some open questions:



Open Question 3

Let $T : E \to Y$ be bounded.

(*i*) $T \in \mathrm{d\text{-}K}(E, Y) \overset{???}{\Longrightarrow} T'' \in \mathrm{d\text{-}K}(E'', Y'')$

(*ii*) $T \in \mathrm{d\text{-}W}(E, Y) \overset{???}{\Longrightarrow} T'' \in \mathrm{d\text{-}W}(E'', Y'')$

(*iii*) $T \in \mathrm{d\text{-}Lim}(E, Y) \overset{???}{\Longrightarrow} T'' \in \mathrm{d\text{-}Lim}(E'', Y'')$

[1] C. D. Aliprantis, O. Burkinshaw, *Positive operators*, Springer, 2006.

[2] J. Bourgain, J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr., **119** (1984), 55–58.

[3] E. Emelyanov, N. Erkurşun-Özcan, S. Gorokhova, *d-Operators in Banach Lattices*, https://arxiv.org/abs/2401.08792v3 (2024).

[4] S. Gorokhova, On compact (limited) operators between Hilbert and Banach spaces, *Filomat*, 38, 11633-11637 (2024).

[5] P. R. Halmos, *A Hilbert Space Problem Book*, Springer, 1982.

Thank you for attention!