$\mathbb{L}$-vector lattices

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Reminder: What is $\mathbb{L}$?

$\mathbb{L}$ is a Dedekind-complete unital f-algebra over $\mathbb{R}$ (in particular, it is a partially ordered ring).
$C(K) \subseteq \mathbb{L} \subseteq C_\infty(K)$ for a Stonean space K.

$\mathbb{R} \subseteq \mathbb{L}$ (the constant functions).
$\mathbb{P} \subseteq \mathbb{L}$ is the set of **idempotents** ($\{0,1\}$-valued functions in $\mathbb{L}$, indicator functions of clopen subsets of K).

Note: if $K = \{*\}$, we get $\mathbb{L} = \mathbb{R}$.

$\mathbb{L}$-vector lattices

An $\mathbb{L}$-vector lattice is a **partially ordered $\mathbb{L}$-module** and a lattice.

Goal: examine how the theory of vector lattices changes when $\mathbb{R}$ is replaced with $\mathbb{L}$.

Notable differences:

- $\mathbb{L}$ is not a field
- $\mathbb{L}$ is not totally ordered
- $\mathbb{L}$ has non-trivial idempotents
- convergence in $\mathbb{L}$ is not topological

A classical theorem

The Riesz-Kantorovich Formulas

If X and Y are $\mathbb{R}$-vector lattices and Y is Dedekind-complete, then

- $\mathcal{L}_{\text{ob}}(X, Y) = \mathcal{L}_{\text{reg}}(X, Y)$ is a Dedekind-complete $\mathbb{R}$-vector lattice, and
- for $S \in \mathcal{L}_{\text{ob}}(X, Y)$, we have
  $$(S \vee 0)(x) = \sup\{S(y) : 0 \leq y \leq x\}$$
  for all $x \in X^+$.

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Extension Lemma

If X and Y are $\mathbb{R}$-vector lattices and Y is Archimedean (e.g. Dedekind-complete), then every additive function $T : X^+ \to Y^+$ extends uniquely to a positive operator $\widehat{T} : X \to Y$ given by $\widehat{T}(x) = T(x^+) - T(x^-)$.

Archimedean $\mathbb{R}$-vector lattices

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For an $\mathbb{R}$-vector lattice Y, these are equivalent to Y being Archimedean:

1. For all $x, y \in Y$, if $\mathbb{N}x \leq y$, then $x \leq 0$.
2. For all $y \in Y^+$, $\inf\left\{\frac{1}{n}y : n \in \mathbb{N}\right\} = 0$.
3. For all $y \in Y^+$, if $D \subseteq \mathbb{R}$ and $\inf_{\mathbb{R}} D = 0$, then $\inf_Y(Dy) = 0$. (Similar for suprema.)
4. For all $y \in Y^+$, if $D \subseteq \mathbb{R}$ has an inf in $\mathbb{R}$, then $\inf_Y(Dy) = (\inf_{\mathbb{R}} D)y$.

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(5. Scalar multiplication $\mathbb{R} \times Y \to Y$ is order-continuous.)

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Y is Archimedean, so $rT(x) \leq T(rx) \leq rT(x)$. Thus $rT(x) = T(rx)$.

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Freudenthal Spectral Theorem

Let $\lambda \in \mathbb{L}^+$. Then there exists a sequence α_n of $\mathbb{Q}$-step functions such that $\alpha_n \uparrow \lambda$.

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If Y is an Archimedean $\mathbb{L}$-vector lattice, and $\lambda_n \uparrow \lambda$ in $\mathbb{L}$, then **we do not necessarily have** $\lambda_n y \uparrow \lambda y$.

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Example

Let $\mathbb{L} = \ell^\infty$ and $Y = \ell^\infty / c_{00}$ (with quotient order). It is easy to show that Y satisfies $\frac{1}{n} y \downarrow 0$ for all $y \in Y^+$.

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$$\lambda_n = (\overbrace{0, \ldots, 0}^{n}, 1, 1, 1, \ldots).$$ Then $\lambda_n \downarrow 0$ in $\mathbb{L}$.

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$\lambda_n = (\overbrace{0, \ldots, 0}^{n}, 1, 1, 1, \ldots)$. Then $\lambda_n \downarrow 0$ in $\mathbb{L}$. Let $y = [(1, 1, \ldots)] \in Y$.
Then $\lambda_n y = [(0, \ldots, 0, 1, 1, \ldots)] = [(1, 1, \ldots)] = y$ for all $n \in \mathbb{N}$.
So $\lambda_n y$ does not decrease to zero in Y.

Notice that $\lambda_n \in \mathbb{P}$ for all $n \in \mathbb{N}$...

Say Y is $\mathbb{P}$-**Archimedean** if

whenever $D \subseteq \mathbb{P}$, $\inf_{\mathbb{L}} D = 0$, and $y \in Y^+$, we have $\inf_Y(Dy) = 0$.

Say Y is $\mathbb{R}$-**Archimedean** if

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If $\mathbb{L} = \mathbb{R}$, then $\mathbb{P} = \{0, 1\}$ and every $\mathbb{L}$-vector lattice is $\mathbb{P}$-Archimedean!

Remarkably, the following are equivalent:

- Y is $\mathbb{R}$-Archimedean and $\mathbb{P}$-Archimedean.
- Whenever $D \subseteq \mathbb{L}$, $\inf_{\mathbb{L}} D = 0$, and $y \in Y^+$, we have $\inf_Y(Dy) = 0$.
- Scalar multiplication $\mathbb{L} \times Y \to Y$ is order-continuous.

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If X and Y are $\mathbb{L}$-vector lattices and Y **is $\mathbb{R}$-Archimedean and $\mathbb{P}$-Archimedean**, then every additive $\mathbb{P}$-**homogeneous** function $T : X^+ \to Y^+$ extends uniquely to a positive linear map $\widehat{T} : X \to Y$ given by $\widehat{T}(x) = T(x^+) - T(x^-)$.

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Same as in the classical case:

- T is order-preserving
- T is $\mathbb{Q}^+$-homogeneous

Combining with $\mathbb{P}$-homegeneity, we get $T(\alpha x) = \alpha T(x)$ for all $x \in X^+$ and all $\mathbb{Q}^+$-step functions $\alpha = \sum_{i=1}^n q_i \pi_i$.
Now $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{L}^+$?

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Freudenthal Spectral Theorem: $\exists$ $\mathbb{Q}^+$-step functions α_n such that $\alpha_n \uparrow \lambda$.
Because $\lambda \in \mathbf{C(K)}$, $\exists$ $\mathbb{Q}^+$-step functions β_n such that $\beta_n \downarrow \lambda$.

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Extension lemma

If X and Y are $\mathbb{L}$-vector lattices and Y is $\mathbb{R}$-Archimedean **and** $\mathbb{P}$-**Archimedean**, then every additive $\mathbb{P}$-**homogeneous** function $T : X^+ \to Y^+$ extends uniquely to a positive operator $\widehat{T} : X \to Y$.

New Riesz-Kantorovich formulas

The Riesz-Kantorovich formulas

If X and Y are $\mathbb{L}$-vector lattices and Y is Dedekind-complete **and** $\mathbb{P}$-**Archimedean**, then

- $\mathcal{L}_{\text{ob}}(X, Y) = \mathcal{L}_{\text{reg}}(X, Y)$ is a Dedekind-complete $\mathbb{L}$-vector lattice, and
- for $S \in \mathcal{L}_{\text{ob}}(X, Y)$, we have
$$(S \vee 0)(x) = \sup\{S(y) : 0 \leq y \leq x\}$$
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For $\pi \in \mathbb{P}$, we have
$$\begin{aligned} T(\pi x) &= \sup\{S(y) : 0 \leq y \leq \pi x\} \\ &= \sup\{S(\pi z) : 0 \leq z \leq x\} \\ &= \sup\{\pi S(z) : 0 \leq z \leq x\} \\ &= \pi T(x). \end{aligned}$$

Aside...

For $y \in Y$, recall:

$$\pi_y := \inf\{\pi \in \mathbb{P} : \pi y = y\}.$$

When is Y **support-attaining**, i.e., when do we have $\pi_y y = y$?

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For Y an $\mathbb{L}$-vector lattice, the following are equivalent:

- Y is $\mathbb{P}$-Archimedean.
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- Every cyclic submodule of Y is projective ($\mathbb{L}y \cong \pi_y \mathbb{L}$).
- Y is a non-singular $\mathbb{L}$-module.

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Examples of support-attaining $\mathbb{L}$-modules:

- $\mathbb{L}$-normed spaces
- Projective $\mathbb{L}$-modules (e.g. free $\mathbb{L}$-modules)
- Any $\mathbb{L}$-module with an essential submodule that is support-attaining
- (Infinite) sums, (infinite) products, and submodules of support-attaining $\mathbb{L}$-modules