

F-decomposable systems

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Positivity XII

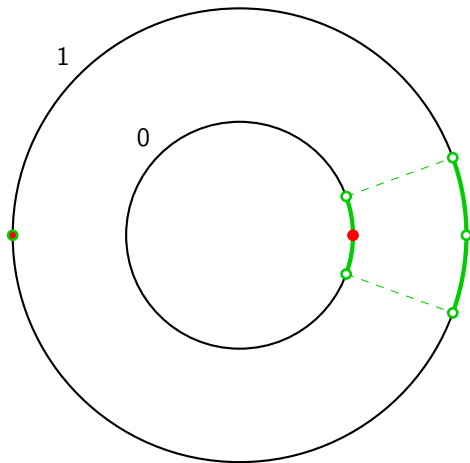
Hammamet, Tunisia

Constructing topological spaces: Resolutions (Fedorchuk ('68))

- Y, X_y for $y \in Y$ - topological spaces.
- $h_y : Y \setminus \{y\} \rightarrow X_y$ - continuous mappings.
- $R(Y, X_y, h_y) := \{\{y\} \times X_y : y \in Y\}$.
- $\{y\} \times V \cup \left(\bigcup \{y' \times X_{y'} : y' \in (U \cap h_y^{-1}(V))\} \right)$ for $U \subset Y, V \subset X_y$ open - basic open sets.

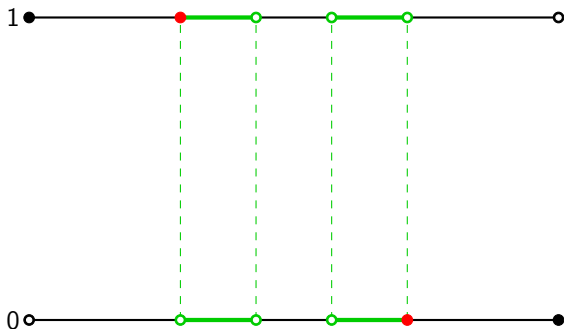
Double circle (Doubling of a compact space)

$$Y = S^1, X_y = \{0, 1\}, h_y(y') = 0 \ \forall y \neq y' \in Y.$$



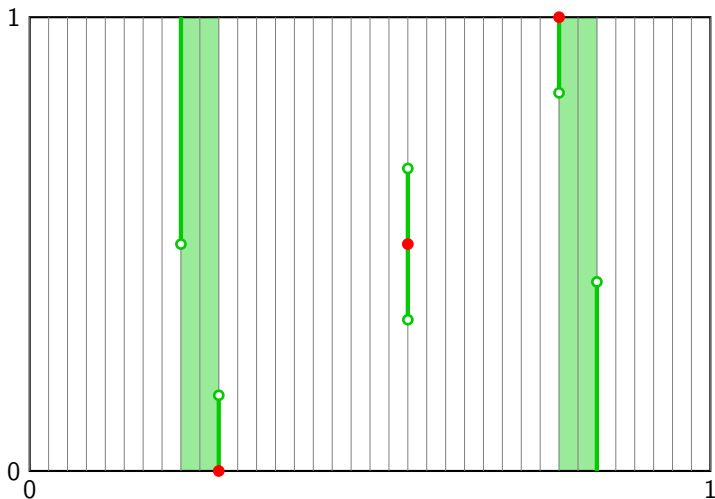
Double arrow (Any separable compact linearly ordered space (Ostraszewski ('74)))

$Y = [0, 1]$, $X_y = \{0, 1\} \forall y \in Y$, $h_y(y') = 0$ if $y' < y$ and $h_y(y') = 1$ if $y' > y$.



Lexicographic square (cube)

$Y = [0, 1]$, $X_y = [0, 1] \forall y \in Y$, $h_y(y') = 0$ if $y' < y$ and $h_y(y') = 1$ if $y' > y$.



Preliminaries

Definition (Fedorchuk ('68))

Let X and Y be Hausdorff compact spaces and $f : X \rightarrow Y$ a continuous onto mapping. f is called *fully closed* if for any two closed disjoint subsets F_1, F_2 of X , the set $f(F_1) \cap f(F_2)$ is finite.

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Proposition

The canonical mapping $\pi : R(Y, X_y, h_y) \rightarrow Y$ (the resolution mapping) is fully closed.

Preliminaries

Definition

Let $S := \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \gamma\}$ be a well-ordered continuous inverse system of Hausdorff compacta with X_0 a singleton. We say that S is an *F-system* if the neighboring bonding mappings, $\pi_\alpha^{\alpha+1}$ for $\alpha + 1 \in \gamma$, are fully closed. We say that S is an *F-decomposable system* if all bonding mappings, π_β^α for $\alpha, \beta \in \gamma$, are fully closed.

Definition (Ivanov ('84))

Let $S := \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \gamma\}$ be an F-system for which the fibers of all neighboring bonding mappings are metrizable. Then the limit $X := \varprojlim S$ is called a *Fedorchuk compact* (*F-compact*) and the length of the system is called its spectral height (denoted $\text{sh}(X)$). If S is moreover F-decomposable, we say that X is an *F_d-compact*.

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Let E be a Banach space. The norm on E is called *locally uniformly rotund (LUR)* if for any point x in the unit sphere S_E and a sequence $\{x_n\}_{n \in \mathbb{N}} \subset S_E$ we have that

$$\lim_{n \rightarrow \infty} \left\| \frac{x + x_n}{2} \right\| = 1 \quad \implies \quad \lim_{n \rightarrow \infty} \|x - x_n\| = 0$$

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Question

For which compact spaces K does $C(K)$ admit an equivalent LUR norm?

Some examples

Theorem (Haydon, Rogers ('90))

Let K be a scattered compact space with $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LUR norm.

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Any scattered compact space is homeomorphic to the limit of an F -decomposable system, with the fibers of neighboring bonding mappings homeomorphic to the one-point compactification of a discrete set.

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Theorem (Haydon, Jayne, Namioka, Rogers ('00))

The space $C([0, 1]_{\text{lex}}^\gamma)$ admits an equivalent LUR (strictly convex) norm if and only if γ is countable.

The problem for F-compacta

Theorem (Gul'ko, Ivanov, Shulikina, Troyanski ('20))

X admits an equivalent pointwise-lower semicontinuous LUR norm whenever X is a Fedorchuk compact of spectral height 3.

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Let X and Y be Hausdorff compacta and π a fully closed mapping from X onto Y . Then $C(X)$ admits an equivalent τ_p -lower semicontinuous LUR norm provided that the spaces $C(Y)$ and $C(\pi^{-1}(y))$ for $y \in Y$ admit equivalent τ_p -lsc LUR norms.

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Theorem (M. (25'))

X admits an equivalent pointwise-lower semicontinuous LUR norm whenever X is an $\mathbb{F}_d$ -compact of countable spectral height.

The main tool

Theorem (Moltó, Orihuela, Troyanski ('97))

Let E be a Banach space and F a norming subspace of its dual. Then E admits an equivalent $\sigma(E, F)$ -lower semicontinuous LUR norm if and only if for any $\epsilon > 0$ there exists a countable decomposition $X = \bigcup_{n \in \mathbb{N}} X_n$ such that for all $n \in \mathbb{N}$ and $x \in X_n$ there exists H an open half space containing x and satisfying:

$$\|\cdot\| \text{-diam}(X_n \cap H) < \epsilon.$$

A characterization (fully closed mapping)

Proposition (Gul'ko, Ivanov, Shulikina, Troyanski ('20))

Let $\pi : X \rightarrow Y$ be a continuous surjective mapping between Hausdorff compacta. Then π is fully closed if and only if for all $f \in C(X)$ we have

$$\left(\operatorname{osc}_{\pi^{-1}(y)} f : y \in Y \right) \in c_0(Y)$$

.

The tree of an inverse system

Let $S := \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \gamma\}$ be a continuous inverse system of Hausdorff compacta.

Define $\Upsilon(S) := \bigcup \{X_\alpha, \alpha \in \gamma\}$ and a partial order $\preceq$ on $\Upsilon(S)$ as follows. If $y \in X_\alpha, x \in X_\beta$, then

$$y \preceq x \iff \alpha \leq \beta \text{ and } \pi_\alpha^\beta(x) = y.$$

We shall refer to $\Upsilon(S)$ as the *tree of the system* S .

A characterization (F-decomposable system)

Lemma

Let S and $\Upsilon(S)$ be as above and $X := \lim_{\leftarrow} S$. Consider the mapping

$$\Phi : C(X) \rightarrow l_{\infty}(\Upsilon(S))$$

$$\Phi(f)(x) := \operatorname{osc}_{\pi_{\alpha}^{-1}(x)} f.$$

Then Φ maps $C(X)$ into $C_0(\Upsilon(S))$ if and only if S is an F -decomposable system.

Another characterization (fully closed mapping)

Notation

Let X and Y be topological spaces and π a continuous mapping from X onto Y . If $M \subset Y$, by Y^M we will denote the quotient space corresponding to the following equivalence classes:

$$[x] = \begin{cases} x, & x \in \pi^{-1}(M); \\ \pi^{-1}(\pi(x)), & \pi(x) \in Y \setminus M. \end{cases}$$

We will denote the corresponding quotient mapping from X to Y^M by p^M and by $\pi^M : Y^M \rightarrow Y$ the unique mapping such that $\pi = \pi^M \circ p^M$.

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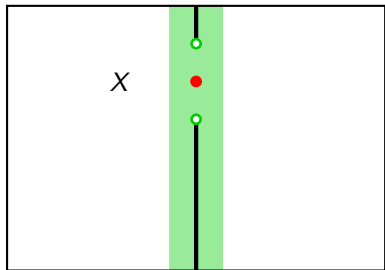
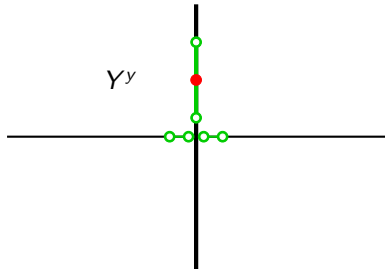
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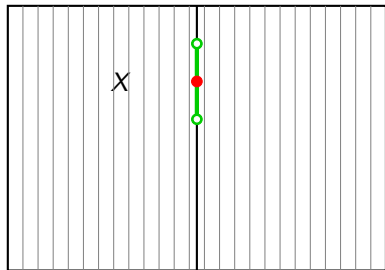
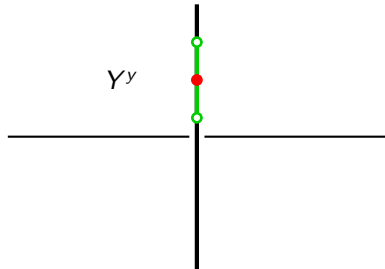
Proposition (Fedorchuk ('06))

Let X and Y be Hausdorff compacta and $f : X \rightarrow Y$ a continuous mapping. Then f is fully closed if and only if for any $M \subset Y$, the space Y^M is Hausdorff.

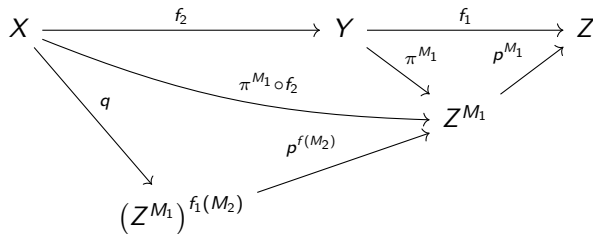
$\mathbb{R}^2$ square



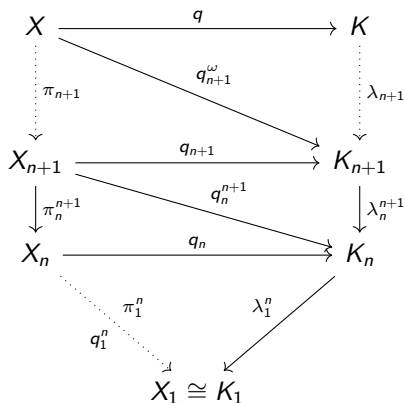
Lexicographic square



Idea of the proof



Idea of the proof



Thank you for the attention!

شكراً

عيدكم مبارك