Vector lattice generated by finite rank operators

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Suppose X is an ordered vector space, Y a subspace of X.
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Idea of proof: $Y \subseteq I_u = C(K)$, use Arzelá–Ascoli Theorem.

Regular operators and Riesz-Kantorovich formulae

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For $T_1, \dots, T_n \in L_r(X, Y)$, their supremum $\bigvee_{i=1}^n T_i$ exists if

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exists for every $x \in X_+$; in this case, $\Big(\bigvee_{i=1}^n T_i\Big)x$ is given by this supremum.

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In this case, $L_r(X, Y)$ is a Banach lattice.

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So the closure $\overline{G(X, Y)}^{\|\cdot\|_r}$ in $L_r(X, Y)$ is the completion of $G(X, Y)$, and is a Banach lattice.

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For $x \in X$, $y \in Y$, we may interpret $x \otimes y$ as a rank-one operator in $L(X^*, Y)$ via $(x \otimes y)(x^*) = x^*(x)y$.

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The two approaches are equivalent because the map $T \mapsto j_Y T$ is a lattice isometry on $G(X^*, Y)$ (with respect to the regular norm).

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This is an alternative definition of $\|\cdot\|_{|\varepsilon|}$.

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Corollary [Arendt,81] $\overline{F(X, Y)}^{\|\cdot\|_r}$ is a Banach lattice.