Isomorphisms of Lattices of Lipschitz Functions

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Thanks

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- T is a $\perp$-isomorphism if it is a $\perp_h$-isomorphism for all $h \in A(X)$.

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Theorem 1 (Leung &T. 2024)

If $\operatorname{Lip}(X,d)$ *is* $\perp$*-isomorphic to* $\operatorname{Lip}(Y,d)$, *then there is a Lipschitz homeomphism* φ *from* (X,ρ) *to* (Y,ρ).

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5. $\mathrm{Lip}^*(X)$ *and* $\mathrm{Lip}(Y)$ *are isomorphic as vector lattices;*
6. $(X, d \wedge 1)$ *and* (Y, ρ) *are Lipschitz homeomorphic.*

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- In this talk, we present the following result:

Theorem 3 (Leung & T. 2025)

Let X be a unbounded closed convex subset of a Banach space E. If $\mathrm{Lip}^*(X)$ and $\mathrm{Lip}(X)$ are $\perp$-isomorphic then X is either a ray or a line.

Proof of Main Result

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Suppose that X is not a ray or a line. Wlog $0 \in X$. According to Theorem 2, if $\mathrm{Lip}^*(X)$ and $\mathrm{Lip}(X)$ are $\perp$-isomorphic, then there is a C-Lipschitz homeomorphism $\varphi : (X, d \wedge 1) \to (X, \rho)$. It can be shown that $\frac{\exp\left(\frac{\|x\|}{2C}\right)}{2} \leq \|\varphi(x)\| \leq e^{2C\|x\|+1}$. In particular, $\|\varphi(x)\| \to \infty$ iff $\|x\| \to \infty$.

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Define $s_X : [1, \infty) \to [0, 2]$ by $s_X(r) = \sup\{\min\{\|u+v\|, \|u-v\|\} : \|u\| = \|v\| = 1, ru, rv \in X\}$. s_X is decreasing. Set $s = \lim_{r\to\infty} s_X(r)$.

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In this case, we have $s = \lim_{r\to\infty} s_X(r) > 0$, where $s_X(r) = \sup\{\min\{\|u+v\|, \|u-v\|\} : \|u\| = \|v\| = 1, ru, rv \in X\}$.

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Fix $c \in (0, s)$. For every $r > 0$, $s_X(r) > c > 0$. Choose $u, v \in X$ so that $\|u\| = \|v\| = 1, ru, rv \in X$ and $\min\{\|u+v\|, \|u-v\|\} > c$. Call the pair of unit vectors $\{u, v\}$ a (c, r)-divergent pair.

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Lemma 4

Suppose that $\lim_{r\to\infty} s_X(r) > c > 0$. *Let* $r \in [1, \infty)$. *For all* $w \in X, \|w\| = r$ *There is a path* γ *in* $X \setminus B(0, \frac{rc}{26})$ *with* $L(\gamma) \le 4r$ *joining* w *and* ru, *where* u *belongs to some* (c, r)*-divergent pair* $\{u, v\}$.

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Recall

$$\frac{\exp\left(\frac{\|x\|}{2C}\right)}{2} \leq \|\varphi(x)\| \leq e^{2C\|x\|+1} - - - (\bigstar)$$

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Let $N \in \mathbb{N}$. By definition of $s_X(N)$, there exists $u, v \in X$, $\|u\| = \|v\| = 1$ so that $Nu, Nv \in X$, $\min\{\|u+v\|, \|u-v\|\} > c > 0$. (i.e., $\{u, v\}$ is a (c, N)-divergent pair).

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So, there exist $a \in [0, Nu]$ and $b \in [0, Nv]$, $\|a\|, \|b\| \ge 1$ so that $\|\varphi(a)\| = \|\varphi(b)\| = \frac{\exp\left(\frac{N}{2C}\right)}{2}$.
Since $\{u, v\}$ is a (c, N)-divergent pair, it can be shown that there is $u^* \in E^*$ so that $\|u^*\| \le \frac{2}{c}$ and $u^*(u) = 1$, $u^*(v) = 0$.

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$$\|a - b\| \ge \frac{|u^*(a-b)|}{\|u^*\|} \ge \frac{c}{2}\|a\|. - - - (\Diamond)$$

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$$\begin{aligned}\|a-b\| &\leq \sum \|x_i - x_{i-1}\| = \sum (\|x_i - x_{i-1}\| \wedge 1)\\ &\leq C \sum \rho(y_{i-1}, y_i) = C \sum \frac{\|y_i - y_{i-1}\|}{\|y_i\| \vee \|y_{i-1}\| \vee 1},\\ &\leq C \frac{\sum \|y_i - y_{i-1}\|}{\frac{c}{26}r} \leq \frac{26C(8r)}{cr} \leq \frac{208C}{c}.\end{aligned}$$

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Hence, by $(\Diamond)$, $\frac{c}{2}\|a\| \leq \frac{208C}{c}$.

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for some constant k. Then by $(\blacktriangle)$,

$$\frac{208C}{c} \geq \frac{c\|a\|}{2} \geq \frac{c}{2}\left(\frac{N}{2C^2} + k\right).$$

This is a contradiction as N can be taken arbitrarily large. (Case 1 proved)

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- What about $\perp$-isomorphism between $\mathrm{Lip}^*(X)$ and $\mathrm{Lip}(Y)$, for different X, Y?

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Theorem 5

Let X, Y be unbounded closed convex subsets of Banach spaces E, F respectively.
(i) If $\lim_{r\to\infty} s_X(r) > 0$, then $\mathrm{Lip}^*(X)$ is not $\perp$-isomorphic to $\mathrm{Lip}(Y)$.
(ii) If $\lim_{r\to\infty} s_X(r) = \lim_{r\to\infty} s_Y(r) = 0$, and $\mathrm{Lip}^*(X)$ $\perp$-isomorphic to $\mathrm{Lip}(Y)$, then X, Y are both lines or rays.

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There are unbounded convex sets $X, Y \subseteq \ell_1$, $\lim_{r\to\infty} s_X(r) = 0$ and $\lim_{r\to\infty} s_Y(r) > 0$ so that $\mathrm{Lip}^*(X)$ is $\perp$-isomorphic to $\mathrm{Lip}(Y)$.

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- Thank you.