Some examples and counterexamples

Youssef AZOUZI

Cathage University, Tunisia

Positivity XII, June 2025, Tunisia

The talk

The talk

I will present some examples and counterexamples related to different types of convergence

The talk

I will present some examples and counterexamples related to different types of convergence

1. Order conbergence
2. Unbounded order convergence
3. Relatively uniform convergence
4. Convegence in T-conditionally probability
5. Unbounded norm convergence.

rp-compactness

Lattice normed spaces.

rp-compactness

Lattice normed spaces.
A triple (E, p, V), where
E is a vector space,
V is an Archimedean vector lattice
$p : E \longrightarrow V$ is a V-vector valued norm.
p-convergence $p(x_\alpha - x) \xrightarrow{o} 0$
rp-convergence $p(x_\alpha - x) \xrightarrow{ru} 0$

rp-compactness

Theorem

An rp-compact operator is sequentially rp-compact.

rp-compactness

Theorem

An rp-compact operator is sequentially rp-compact.

Problem

What about the converse?

rp-compactness

sequential rp-compactness does not imply rp-compactness.

rp-compactness

sequential rp-compactness does not imply rp-compactness.

Example

Let $F = \left\{x \in \mathbb{R}^{\mathbb{R}} : Supp\left(x\right) \text{ is countable}\right\}$ and

$$P : E = F \oplus \mathbb{R}\mathbf{1}. \longrightarrow F; \qquad x + r\mathbf{1} \longmapsto x.$$

1. P is sequentially rp-compact.
   1. Assume that $|f_n| = |g_n + \lambda_n \mathbf{1}| \leq f = g + \lambda \mathbf{1} \in E^+$, $n \in \mathbb{N}$.
   2. Then $|\lambda_n| \leq \lambda$.
   3. Diagonal process $\Longrightarrow \exists \left(h_n\right) = \left(g_{\varphi(n)}\right) : \left(h_n\right)$ converges pointwise on A.
   4. We get $h_n \xrightarrow{ru} h$ in F.

rp-compactness

sequential rp-compactness does not imply rp-compactness.

Example

Let $F = \left\{ x \in \mathbb{R}^{\mathbb{R}} : Supp\left(x\right) \text{ is countable} \right\}$ and

$$P : E = F \oplus \mathbb{R}\mathbf{1}. \longrightarrow F; \qquad x + r\mathbf{1} \longmapsto x.$$

1. P is sequentially rp-compact.
   1. Assume that $|f_n| = |g_n + \lambda_n \mathbf{1}| \leq f = g + \lambda \mathbf{1} \in E^+$, $n \in \mathbb{N}$.
   2. Then $|\lambda_n| \leq \lambda$.
   3. Diagonal process $\Longrightarrow \exists \left(h_n\right) = \left(g_{\varphi(n)}\right)$: $\left(h_n\right)$ converges pointwise on A.
   4. We get $h_n \xrightarrow{ru} h$ in F.
2. P is not bounded :
   The net $\left(u_\alpha\right) = \left(\chi_{\{\alpha\}}\right)_{\alpha \in \mathbb{R}}$ is bounded in E and $Tu_\alpha = u_\alpha$ is not bounded in F.

Convergence in probability

Convergence in probability

- $X_n \xrightarrow{\mathbb{P}} X : \mathbb{P}\left(|X_n - X| \geq \varepsilon\right) \longrightarrow 0 \ \forall \varepsilon > 0$
- Or equivalently $\mathbf{E}\left(\mathbf{1}_{(|X_n - X| - \varepsilon \mathbf{1})^+ > 0}\right) \longrightarrow 0$
- It becomes

$$TP_{(|X_n - X| - \varepsilon e)^+} e \longrightarrow 0 \text{ in order.}$$

- With Grobler's notations :

$$\mathbb{P}\left(|x_n - x| \geq \varepsilon e\right) \xrightarrow{o} 0.$$

Unbounded order convergence probability

Unbounded order convergence probability

A result we want to generalize

Theorem

If $X_n \longrightarrow X$ in probability then there exists a subsequence $X_{\varphi(n)}$ which converges to X a.s.

Unbounded order convergence probability

A result we want to generalize

Theorem

If $X_n \longrightarrow X$ in probability then there exists a subsequence $X_{\varphi(n)}$ which converges to X a.s.

Possible statements

1. Given a conditional Riesz triple (E, T, e):
   If $x_n \longrightarrow x$ in T-conditionally probability then for some subsequence, $x_{\varphi(n)} \xrightarrow{uo} x$.
2. Analogue question for nets.

Negative answers

Negative answers

Example

Let $E = \left(L^1\right)^\Gamma$ where Γ is the set of strictly increasing maps from $\mathbb{N}$ to $\mathbb{N}$. $T : E \longrightarrow E$; $\left(f_\varphi\right) \longmapsto \left(\left(\int f_{\varphi f}\right) \mathbf{1}\right)$.

1. Pick a sequence (v_n) in L^1 s.t. $v_n \xrightarrow{\mathbb{P}} 0$ but v_n does not cv a.s.
2. let $y^\varphi_{\varphi(n)} = v_n$ and $y^\varphi_k = 0$ for the other values of k.
3. Finally define $x_n \in E$ by : $x_n = \left(y^\varphi_n\right)_{\varphi\in\Gamma}$.

Negative answers

Example

Let $E = \left(L^1\right)^\Gamma$ where Γ is the set of strictly increasing maps from $\mathbb{N}$ to $\mathbb{N}$. $T : E \longrightarrow E$; $\left(f_\varphi\right) \longmapsto \left(\left(\int f_{\varphi f}\right) \mathbf{1}\right)$.

1. Pick a sequence $\left(v_n\right)$ in L^1 s.t. $v_n \xrightarrow{\mathbb{P}} 0$ but v_n does not cv a.s.
2. let $y^\varphi_{\varphi(n)} = v_n$ and $y^\varphi_k = 0$ for the other values of k.
3. Finally define $x_n \in E$ by : $x_n = \left(y^\varphi_n\right)_{\varphi \in \Gamma}$.
4. Then $\left(x_n\right)$ converges to 0 in T-conditionally probability. However for every extraction ψ we have $\left(x_{\psi(n)}\left(\psi\right)\right)_N = \left(v_n\right)_N$ is not uo-convergent.

Case of nets

Case of nets

We come back to the second question for nets

2a. (Even in the classical case)
If $x_\alpha \longrightarrow x$ in T-conditionally probability then for some subnet $y_\beta \xrightarrow{uo} x$.

2b. In a Banach lattice, if If $x_\alpha \longrightarrow x$ in norm then $y_\beta \xrightarrow{o} x$. for some subnet.

2c. In a Banach lattice, if If $x_\alpha \longrightarrow x$ in norm then $y_\beta \xrightarrow{uo} x$. for some subnet.

Negative answer
A first example

Negative answer
A first example

Example

$X = \{x \in \ell_\infty(\mathbb{R}) \text{ with countable support}\}$.

Consider the net (x_α) defined by $x_\alpha = \frac{1}{|\alpha|}\chi_\alpha$, $\alpha \in \mathcal{P}_f(\mathbb{R})$.

Then $x_\alpha \xrightarrow{\|.\|} 0$ but has no order convergent subnet.

Negative answer
A second example

Negative answer
A second example

Example

For $S \subseteq [0,1]$ finite and $\varepsilon > 0$ let $V_{(S,\varepsilon)}$ be an open set : $S \subseteq V$ and $\mu(S) \leq \varepsilon$. Put $f_{(S,\varepsilon)} = \chi_{V_{(S,\varepsilon)}}$.

- We write $\alpha = (S,\varepsilon) \leq \alpha' = (S',\varepsilon')$ if $S \subset S'$ and $\varepsilon \geq \varepsilon'$.
- Then we have $f_\alpha \xrightarrow{\|.\|} 0$ in $L^1[0,1]$.
- But (f_α) has no order (unbounded order) cv subnet.
  1. Assume that $\left(f_{\varphi(\beta)}\right) \xrightarrow{o} 0$. Then $f_{\varphi(\beta)} \leq g_\beta$ eventually for some $g_\beta \downarrow 0$. Fix $\beta_0 \in B$ and $(S,\varepsilon) = \varphi(\beta_0)$. For every $x \in [0,1]$, there is $\beta_x \in B$ such that $\beta_x \geq \beta_0$ and $\varphi(\beta_x) \geq (S \cup \{x\}, \varepsilon)$. Let $V_x = V_{\varphi(\beta_x)}$ then $f_{\varphi(\beta_x)} = \chi_{V_x} \leq g_{\beta_x} \leq g_{\beta_0}$.*By* compactness $[0,1] = \bigcup_{k=1}^{n} V_{x_k}$.
  2. Hence $g_{\beta_0} \geq 1 = \chi_{[0,1]}$. $(\forall \beta_0)$
  3. A contradiction!

Completeness

Completeness

Usual definitions and notations

- E is **Dedekind complete** if ...

Completeness

Usual definitions and notations

- E is **Dedekind complete** if ...
- E is uniformly complete if ....

Completeness

Usual definitions and notations

- E is **Dedekind complete** if ...
- E is uniformly complete if ....
- E^{ru} is the relatively uniform completion of E.

Completeness

Usual definitions and notations

- E is **Dedekind complete** if ...
- E is uniformly complete if ....
- E^{ru} is the relatively uniform completion of E.
- E^{δ} is the Dedekind completion of E.

Completeness

Usual definitions and notations

- E is **Dedekind complete** if ...
- E is uniformly complete if ....
- E^{ru} is the relatively uniform completion of E.
- E^{δ} is the Dedekind completion of E.
- It is well-known that E^{ru} is the intersection of all uniformly complete vector sublattices of E^{δ} that contain E.

Completion of a principal ideal

Completion of a principal ideal

- E_x is the ideal generated by x in E.

Completion of a principal ideal

- E_x is the ideal generated by x in E.
- It is easy to see that $\left(E^{\delta}\right)_x$ can be naturally identified with $(E_x)^{\delta}$. That is
$$\left(E^{\delta}\right)_x = (E_x)^{\delta}.$$

Completion of a principal ideal

- E_x is the ideal generated by x in E.
- It is easy to see that $\left(E^{\delta}\right)_x$ can be naturally identified with $(E_x)^{\delta}$. That is

$$\left(E^{\delta}\right)_x = (E_x)^{\delta}.$$

Completion of a principal ideal

- E_x is the ideal generated by x in E.
- It is easy to see that $\left(E^{\delta}\right)_x$ can be naturally identified with $(E_x)^{\delta}$. That is

$$\left(E^{\delta}\right)_x = (E_x)^{\delta}.$$

Problem

What about ru-completion?

Completion of a principal ideal

- E_x is the ideal generated by x in E.
- It is easy to see that $\left(E^\delta\right)_x$ can be naturally identified with $(E_x)^\delta$. That is

$$\left(E^\delta\right)_x = (E_x)^\delta .$$

Problem

What about ru-completion?

Is it true that

$$(E^{ru})_x = (E_x)^{ru}?$$

The answer is

The answer is NO

The answer is NO

Example

Let E be the space of linear piecewise and continuous functions on $[0,1]$
The $E^{ru} = C[0,1]$. Hence

1. Define $u \in E$ by $u(t) = t$. Then

The answer is NO

Example

Let E be the space of linear piecewise and continuous functions on $[0,1]$
The $E^{ru} = C[0,1]$. Hence

1. Define $u \in E$ by $u(t) = t$. Then
2. $(E_u)^{ru} = \{f \in E : \exists \varphi_n \in E_u, \varepsilon_n > 0 : |\varphi_n - f| \leq \varepsilon_n u, \varepsilon_n \longrightarrow 0\}$.

The answer is NO

Example

Let E be the space of linear piecewise and continuous functions on $[0,1]$
The $E^{ru} = C\,[0,1]$. Hence

1. Define $u \in E$ by $u(t) = t$. Then
2. $(E_u)^{ru} = \{f \in E : \exists \varphi_n \in E_u, \varepsilon_n > 0 : |\varphi_n - f| \leq \varepsilon_n u, \varepsilon_n \longrightarrow 0\}$.
3. The inclusion $(E_u)^{ru} \subset (E^{ru})_u$ is strict.

Proof

Proof

Put $H = \{f \in E : \exists \varphi_n \in E_u, \varepsilon_n > 0 : |\varphi_n - f| \leq \varepsilon_n u, \varepsilon_n \longrightarrow 0\}$.

Proof.

- Clearly $H \subseteq (E_u)^{ru}$.
- For the converse it is enough to prove that H is ru complete.
- Assume that (f_n) is ru-Cauchy in H.
- Then $\exists (\varepsilon_n) \downarrow 0$, s.t. $|f_n - f_m| \leq \varepsilon_n u, \qquad$ for $m \geq n \geq 1$.
- Let $f = \lim f_n$ in $E^{ru} = C[0,1]$.
- We get $|f_n - f| \leq \varepsilon_n u, \qquad$ for $n \geq 1$.
- As $f_n \in H$ $\exists \varphi_n \in E_u$ such $|\varphi_n - f_n| \leq \frac{1}{n} u$.
- Hence $|\varphi_n - f| \leq \left(\frac{1}{n} + \varepsilon_n\right) u$,for all $n \in \mathbb{N}$. Thus $f \in H$.
- Thus $f_n \xrightarrow{ru} f$ in H.
- This shows that H is r.u. complete as required.

The proof (Cont.)

The proof (Cont.)

The inclusion is strict

Proof.

- Consider tow real sequences (a_n) and (b_n) satisfying :
  (i) $0 < a_{n+1} < b_{n+1} < a_n < b_n < .... < a_1 < b_1 = 1$, and
  (ii) $\lim b_n = 0 = \lim a_n$.
- Define a function f on $[0,1]$ by putting $f(0) = 0$, $f(b_n) = 0$, $f(a_n) = a_n$ for $n \in \mathbb{N}$, and f is linear on each of the intervals $[a_n, b_n], [b_n, a_{n-1}]$.

□

The proof (Cont.)

The proof (Cont.)

Proof.

- Observe that $f \in E$ and $0 \leq f \leq u$. Hence $f \in \left(E^{ru}\right)_u$.
- **But** $f \notin H = \left(E_u\right)^{ru}$? Assume $f \in \left(E_u\right)^{ru}$.

1. Then $\left|\varphi_n - f\right| \leq \varepsilon_n u \qquad n \in \mathbb{N}, \qquad$ Here $\left(\varphi_n\right) \subseteq E$ and $\varepsilon_n \downarrow 0$.
2. Pick an integer n_0 such that $\varepsilon_{n_0} \leq 1/3$. As $\varphi_{n_0} \in E_u$ there exist $\delta > 0, \lambda \in \mathbb{R}$ such that $\varphi_{n_0}(t) = \lambda t$ for $t \in [0, \delta]$. Thus

$$\left|\lambda t - f(t)\right| \leq 1/3.t \text{ for } t \in [0, \delta].$$

3. Applying this to $t = b_k$ for k large enough yields

$$|\lambda| \leq 1/3.$$

4. Applying for $t = a_k$ for k large enough we get

$$|\lambda - 1| \leq 1/3.$$

These last two inequalities are incompatible.

Two more examples

De la Vallé Poussin Theorem

De la Vallé Poussin Theorem

De La Vallé Poussin's classical theorem offers a characterization of uniform integrability.

Theorem

Let $\mathcal{H}$ be a family in $L^1(\mathbb{P})$. Then the following statements are equivalent:

(i) *The family $\mathcal{H}$ is uniform integrable;*

(ii) *There exists a convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that*

$$\lim_{x\longrightarrow\infty} \frac{\varphi(x)}{x} = \infty \text{ and } \sup_{X\in\mathcal{H}} \mathbb{E}\left(\varphi\left(|X|\right)\right) < \infty.$$

De la Vallé Poussin Theorem

De La Vallé Poussin's classical theorem offers a characterization of uniform integrability.

Theorem

Let $\mathcal{H}$ be a family in $L^1(\mathbb{P})$. Then the following statements are equivalent:

(i) *The family $\mathcal{H}$ is uniform integrable;*

(ii) *There exists a convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that*

$$\lim_{x \longrightarrow \infty} \frac{\varphi(x)}{x} = \infty \text{ and } \sup_{X \in \mathcal{H}} \mathbb{E}(\varphi(|X|)) < \infty.$$

Problem

Is it true in the ontext of Riesz spaces?

De la Vallé Poussin Theorem

De la Vallé Poussin Theorem

We consider a Riesz conditional triple (E, T, e)

Example

Let E be the Riesz space consisting of all functions $f : [1, \infty[\longrightarrow \mathbb{R}$ such that $f(x) = O(x)$ as $x \longrightarrow \infty$.
E is a Dedekind complete Riesz space with weak order unit e, where $e(t) = 1$ for all $t \in [1, \infty)$. Let $T : E \longrightarrow E$ be the identity map and $\mathcal{F} = \{u\}$ with $u(x) = x$. Then T is a conditional expectation and $\mathcal{F}$ is T-uniform. If φ is a test function, which means that φ is a convex function on $[1, \infty)$ and $\lim \dfrac{\varphi(x)}{x} = \infty$) then $\varphi(u) \in E^s \diagdown E$.

De la Vallé Poussin Theorem

We consider a Riesz conditional triple (E, T, e)

Example

Let E be the Riesz space consisting of all functions $f : [1, \infty[\longrightarrow \mathbb{R}$ such that $f(x) = O(x)$ as $x \longrightarrow \infty$.
E is a Dedekind complete Riesz space with weak order unit e, where $e(t) = 1$ for all $t \in [1, \infty)$. Let $T : E \longrightarrow E$ be the identity map and $\mathcal{F} = \{u\}$ with $u(x) = x$. Then T is a conditional expectation and $\mathcal{F}$ is T-uniform. If φ is a test function, which means that φ is a convex function on $[1, \infty)$ and $\lim \dfrac{\varphi(x)}{x} = \infty$) then $\varphi(u) \in E^s \diagdown E$.

Some hope

De la Vallé Poussin Theorem

We consider a Riesz conditional triple (E, T, e)

Example

Let E be the Riesz space consisting of all functions $f : [1, \infty[\longrightarrow \mathbb{R}$ such that $f(x) = O(x)$ as $x \longrightarrow \infty$.
E is a Dedekind complete Riesz space with weak order unit e, where $e(t) = 1$ for all $t \in [1, \infty)$. Let $T : E \longrightarrow E$ be the identity map and $\mathcal{F} = \{u\}$ with $u(x) = x$. Then T is a conditional expectation and $\mathcal{F}$ is T-uniform. If φ is a test function, which means that φ is a convex function on $[1, \infty)$ and $\lim \dfrac{\varphi(x)}{x} = \infty$) then $\varphi(u) \in E^s \diagdown E$.

Some hope

- Note that $\varphi(u) \in E^u$, the universal completion of E, and even better $\varphi(u) \in L^1(T)$.

De la Vallé Poussin Theorem

We consider a Riesz conditional triple (E, T, e)

Example

Let E be the Riesz space consisting of all functions $f : [1, \infty[\longrightarrow \mathbb{R}$ such that $f(x) = O(x)$ as $x \longrightarrow \infty$.
E is a Dedekind complete Riesz space with weak order unit e, where $e(t) = 1$ for all $t \in [1, \infty)$. Let $T : E \longrightarrow E$ be the identity map and $\mathcal{F} = \{u\}$ with $u(x) = x$. Then T is a conditional expectation and $\mathcal{F}$ is T-uniform. If φ is a test function, which means that φ is a convex function on $[1, \infty)$ and $\lim \dfrac{\varphi(x)}{x} = \infty$) then $\varphi(u) \in E^s \diagdown E$.

Some hope

- Note that $\varphi(u) \in E^u$, the universal completion of E, and even better $\varphi(u) \in L^1(T)$.
- Thus (ii) is indeed fulfilled if we work in the space $L^1(T)$.

De la Vallé Poussin Theorem

We consider a Riesz conditional triple (E, T, e)

Example

Let E be the Riesz space consisting of all functions $f : [1, \infty[\longrightarrow \mathbb{R}$ such that $f(x) = O(x)$ as $x \longrightarrow \infty$.
E is a Dedekind complete Riesz space with weak order unit e, where $e(t) = 1$ for all $t \in [1, \infty)$. Let $T : E \longrightarrow E$ be the identity map and $\mathcal{F} = \{u\}$ with $u(x) = x$. Then T is a conditional expectation and $\mathcal{F}$ is T-uniform. If φ is a test function, which means that φ is a convex function on $[1, \infty)$ and $\lim \dfrac{\varphi(x)}{x} = \infty$) then $\varphi(u) \in E^s \setminus E$.

Some hope

- Note that $\varphi(u) \in E^u$, the universal completion of E, and even better $\varphi(u) \in L^1(T)$.
- Thus (ii) is indeed fulfilled if we work in the space $L^1(T)$.
- One might then expect that Theorem 10 holds in the general

A negative answer

A negative answer

Example

Let A denotes the set of all convex functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying $\lim_{x \longrightarrow \infty} \frac{\varphi(x)}{x} = \infty$ and consider the conditional Riesz triple (E, T, e) with $E = \left(L^1[0,1]\right)^A$, $e = \left(e_\varphi = 1\right)_{\varphi \in A}$ and T is defined as follows

$$Tf = \left(\int f_\varphi d\mu\right)_{\varphi \in A} \text{ for all } f = \left(f_\varphi\right)_{\varphi \in A} \in E.$$

A negative answer

Example

Let A denotes the set of all convex functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying $\lim_{x \longrightarrow \infty} \frac{\varphi(x)}{x} = \infty$ and consider the conditional Riesz triple (E, T, e) with $E = \left(L^1[0,1]\right)^A$, $e = \left(e_\varphi = 1\right)_{\varphi \in A}$ and T is defined as follows

$$Tf = \left(\int f_\varphi d\mu\right)_{\varphi \in A} \text{ for all } f = \left(f_\varphi\right)_{\varphi \in A} \in E.$$

Now for each $\varphi \in A$ one can find an element $X_\varphi \in L^1[0,1]$ such that

$$\int_{\{|X_\varphi| \geq n\}} |X_\varphi| \, d\mu \leq \frac{1}{n} \text{ for all } n = 1, 2, \ldots \text{ and } \int \varphi\left(|X_\varphi|\right) d\mu = \infty.$$

A negative answer

Example

Let A denotes the set of all convex functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\lim\limits_{x\longrightarrow\infty} \frac{\varphi(x)}{x} = \infty$ and consider the conditional Riesz triple (E, T, e) with $E = \left(L^1[0,1]\right)^A$, $e = \left(e_\varphi = 1\right)_{\varphi\in A}$ and T is defined as follows

$$Tf = \left(\int f_\varphi d\mu\right)_{\varphi\in A} \text{ for all } f = \left(f_\varphi\right)_{\varphi\in A} \in E.$$

Now for each $\varphi \in A$ one can find an element $X_\varphi \in L^1[0,1]$ such that

$$\int_{\{|X_\varphi|\geq n\}} |X_\varphi|\, d\mu \leq \frac{1}{n} \text{ for all } n = 1, 2, \ldots \text{ and } \int \varphi\left(|X_\varphi|\right) d\mu = \infty.$$

Furthermore let $Y^\varphi \in L^1(T)$ be defined as follows: $Y^\varphi(\psi) = X_\varphi$ if $\psi = \varphi$ and $Y^\varphi(\psi) = 0$ otherwise. The family $\left(Y^\varphi\right)_{\varphi\in A}$ is T-uniform. But there is no function φ in A such that $\sup T\varphi(|Y|)$ exists in $L^1(T)$.

Thank you